Mader's conjecture on subdivision of digraphs

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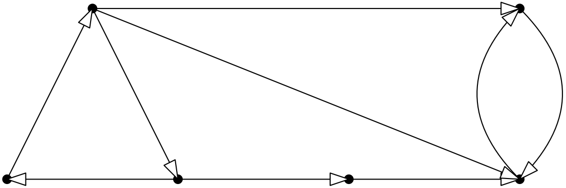
Subdivision of a digraph



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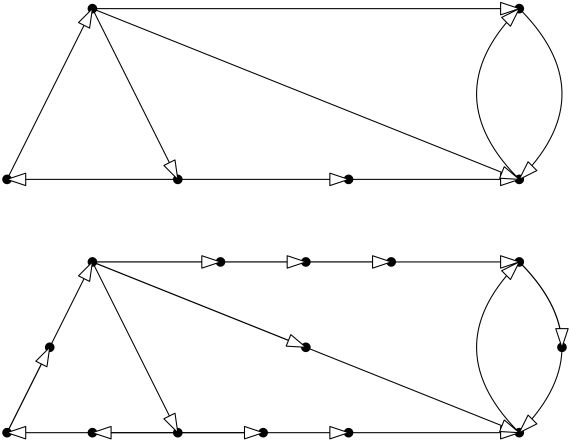

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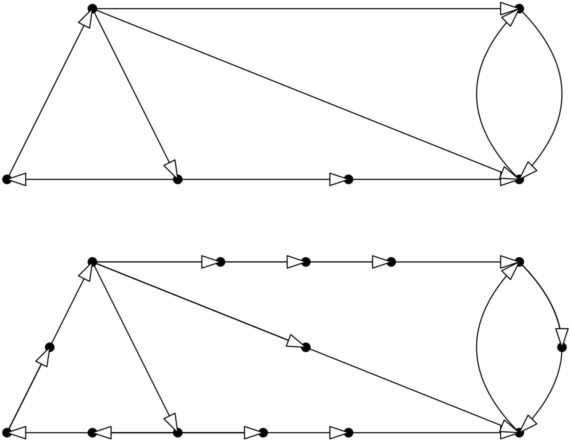

The subdivision of undirected graphs is similar

Motivation

Theorem [Mader, 1967]

There exists an integer $f(k)$ such that every graph with minimum degree at least $f(k)$ contains a subdivision of the clique on k vertices

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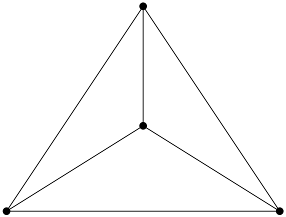

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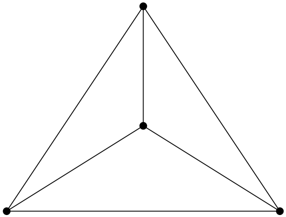

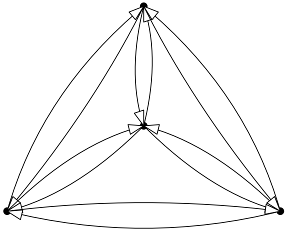

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This conjecture is **false**

Motivation

Theorem [Thomassen, 1986]

There exists a class of digraphs with arbitrarily large minimum in- and out-degree and no directed cycle of even length

Every subdivision of $\vec{K}_3$ contains an even cycle

Motivation

Conjecture

Let G be an acyclic digraph. There exists an integer $f(G)$ such that every digraph D with $\delta^+(D) \geq f(G)$ contains a subdivision of G

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It was proved for $k \leq 4$ by Mader

A weaker conjecture

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Theorem

Let T be an in-arborescence. There exists an integer $f(T)$ such that every digraph D with $\delta^+(D) \geq f(T)$ contains a subdivision of T

What about digraphs containing cycles?

Fact

Every digraph D has a cycle of size at least $\delta^+(D) + 1$ (i.e. a subdivision of C_2)

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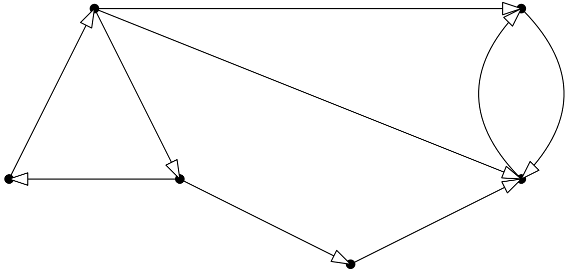
Conjecture [Bermond and Thomassen, 1981]

Every digraph D with $\delta^+(D) \geq 2k-1$ has k disjoint directed cycles

Strongly connected components

- **Strongly connected** if for every $u, v \in V(D)$ there exists a directed path from u to v
- **Strongly connected component** is a maximal strongly connected subgraph



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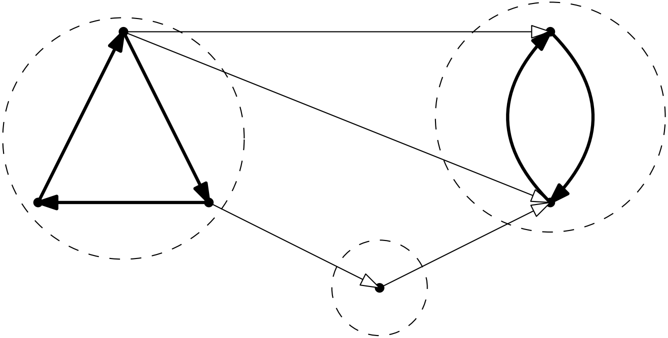

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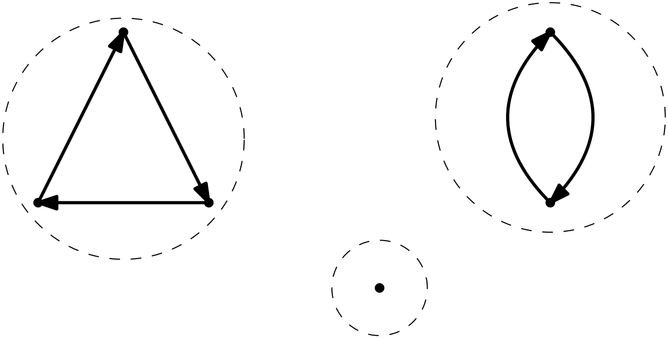
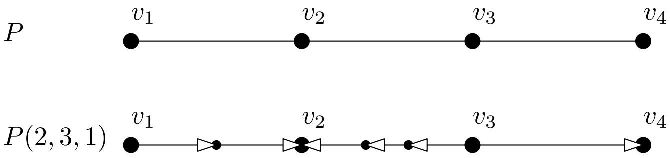


Subdivision of directed paths

- Let $P = (v_1 v_2 \cdots v_n)$ be a path
- $P(k_1, k_2, \ldots, k_{n-1})$ is obtained from P by replacing every edge $\{v_i, v_{i+1}\}$ by a directed path of length k_i from v_i to v_{i+1} if i is odd, and from v_{i+1} to v_i if i is even



Subdivision of directed paths

Theorem [Aboulker, Cohen, Havet, Lochet, M., and Thomassé, 2016+]

Let $P(k_1, k_2, \dots, k_\ell)$ be a path, and let D be a digraph with $\delta^+(D) \geq \sum_{i=1}^{\ell} k_i$. For every $v \in V(D)$, D contains a path $P(k'_1, k'_2, \dots, k'_\ell)$ with initial vertex v such that $k'_i \geq k_i$ if i is odd, and $k'_i = k_i$ otherwise

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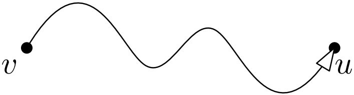
Proof

By induction on ℓ. For every path $P(x_1, x_2, \dots, x_t)$ with $t < \ell$ and every digraph G with $\delta^+(G) \geq \sum_{i=1}^{t} x_i$, the result holds

Induction step



- $\delta^+(D) \geq \sum_{i=1}^{\ell} k_i \implies \exists\ P_{v,u}$ of length k_1

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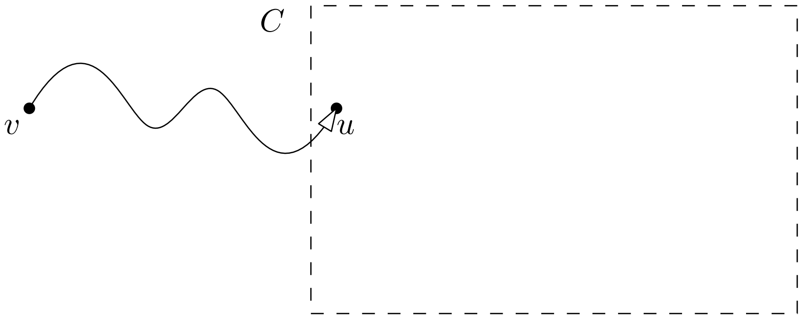

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- C is the component of $D - (P_{v,u} - u)$ containing u

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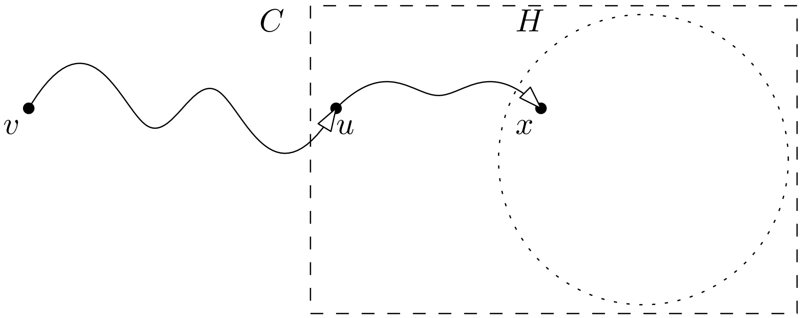

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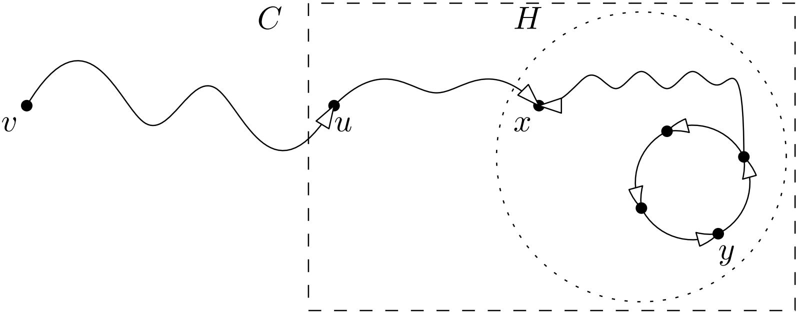

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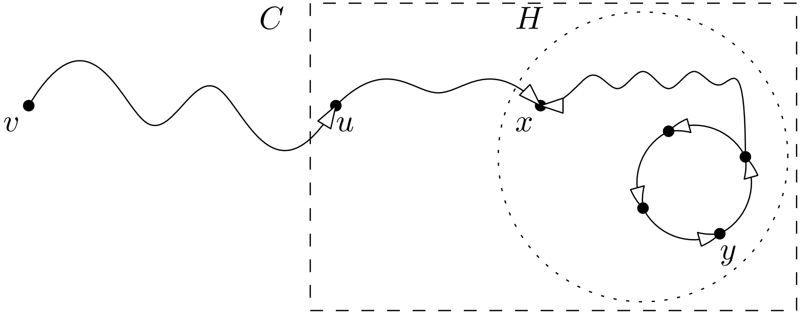

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- C is the component of $D - (P_{v,u} - u)$ containing u
- H strongly connected and $\delta^+(H) \geq \delta^+(D) - k_1 \implies \exists\ P_{y,x}$ of length k_2
- Apply induction hypothesis on $G := H - (P_{y,x} - y)$ (note that $\delta^+(G) \geq \delta^+(D) - k_1 - k_2$)

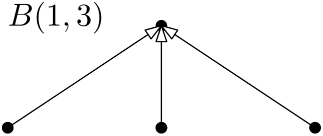
in-Arborescences

The ℓ-**branching in-arborescence of depth** k $(B(k,\ell))$:

- $B(0,\ell)$ is a single vertex, which is the root and the leaf of $B(0,\ell)$
- $B(k,\ell)$ is obtained from $B(k-1,\ell)$ by taking each leaf of $B(k-1,\ell)$ and adding ℓ new vertices dominating this leaf

Let $b(k,\ell) := |V(B(k,\ell)|$; so $b(k,\ell) = \sum_{i=0}^{k} \ell^i = \frac{1-\ell^{k+1}}{1-\ell}$

Examples of in-arborescences



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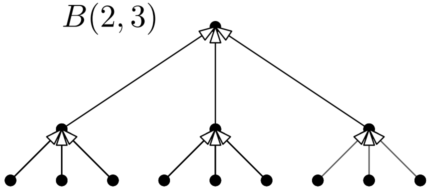

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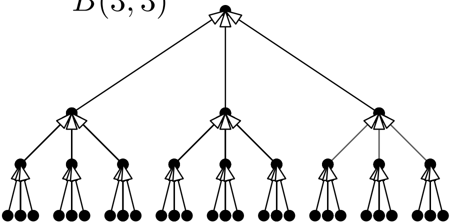


Recall the theorem

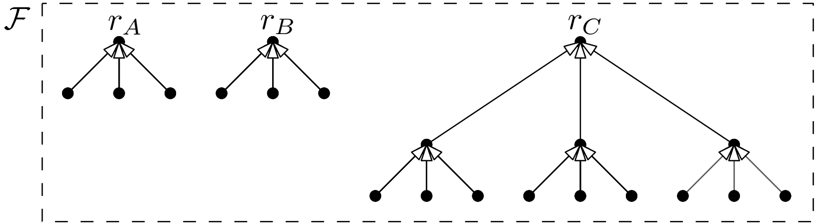
Theorem [Aboulker, Cohen, Havet, Lochet, M., and Thomassé, 2016+]

There exists an integer $f(k, \ell)$ such that every digraph D with $\delta^+(D) \geq f(k, \ell)$ contains a subdivision of $B(k, \ell)$

Proof idea

- By induction on k and ℓ.
- Start with a packing of ℓ-**branching** in-arborescences that covers a maximum number of vertices

Proof for in-arborescences



- $\mathcal{F}$ is a packing of in-arborescence that covers a maximum number of vertices

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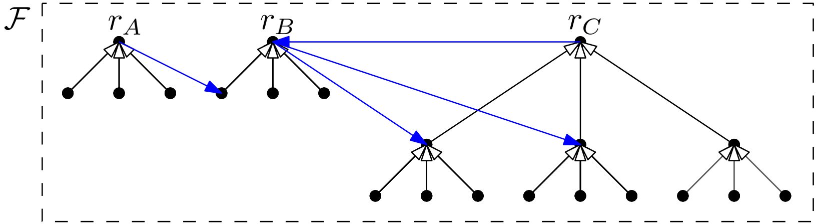

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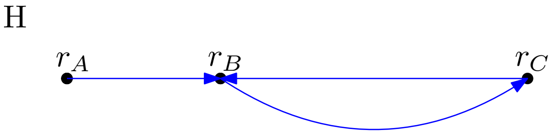

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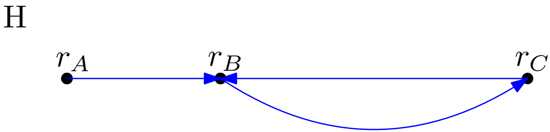
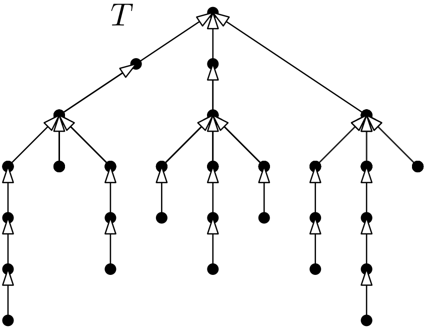


- $\mathcal{F}$ is a packing of in-arborescence that covers a maximum number of vertices
- H is reduced graph of roots
- Suppose that $\delta^+(H) \geq f(k-1, p)$, where p is **large** compared to ℓ ($p := b(k-1,\ell) \cdot (\ell - 1) + 1$)

Case 1: $\delta^+(H) \geq f(k-1, p)$

By the induction hypothesis, there exists a subdivision of $B(k-1, p)$ in H



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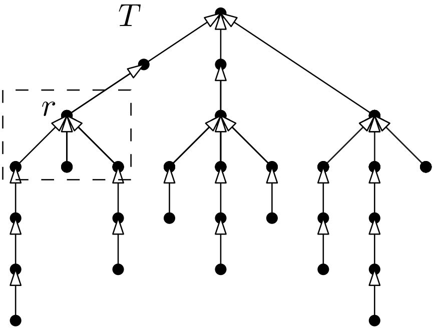
Consider a branching vertex r and the p vertices that dominate r in T



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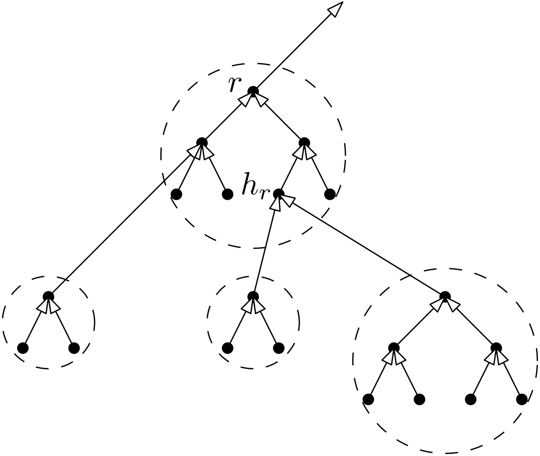
Because $p = b(k-1,\ell)\cdot(\ell-1)+1$, there exists a vertex in the arborescence rooted at r which is dominated by ℓ vertices in D



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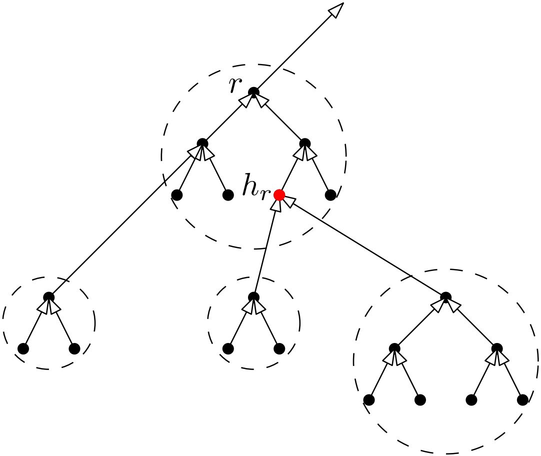

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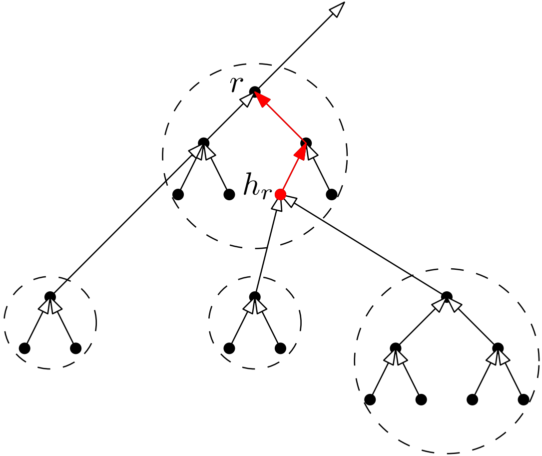

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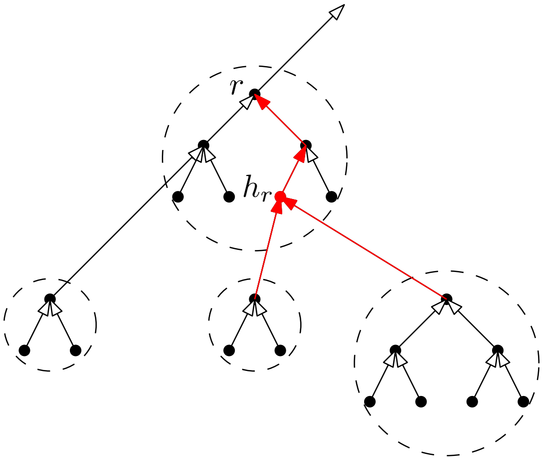

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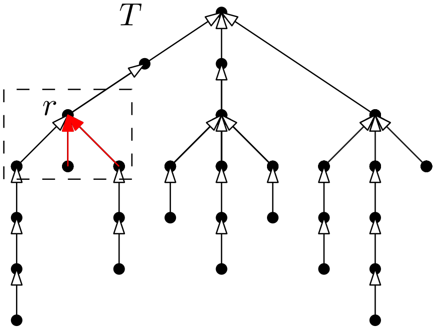
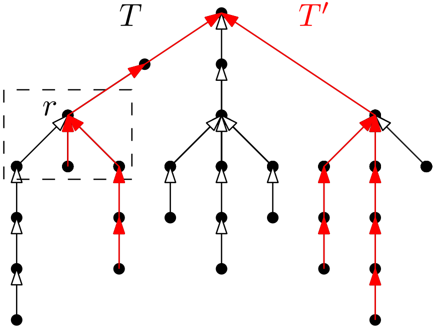
A similar construction holds for every non-leaf vertex of T



Case 1: $\delta^+(H) \geq f(k-1,p)$

Recall T is a subdivision of $B(k-1,p)$ in H. Then, $T' \subset T$ corresponds to a subgraph of D that contains a subdivision of $B(k,\ell)$



Case 2: $\delta^+(H) < f(k-1, p)$

- We show a "Menger-like" theorem to prove the existence of vertex-disjoint directed paths

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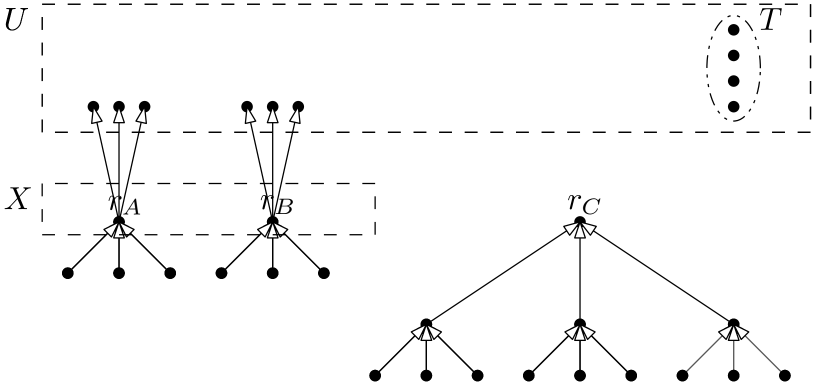

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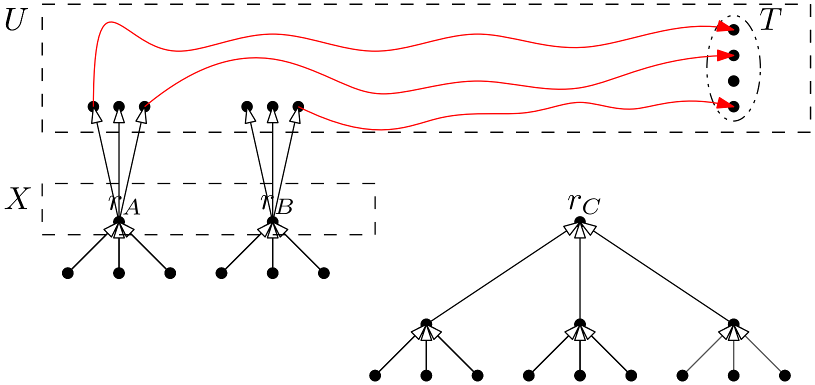

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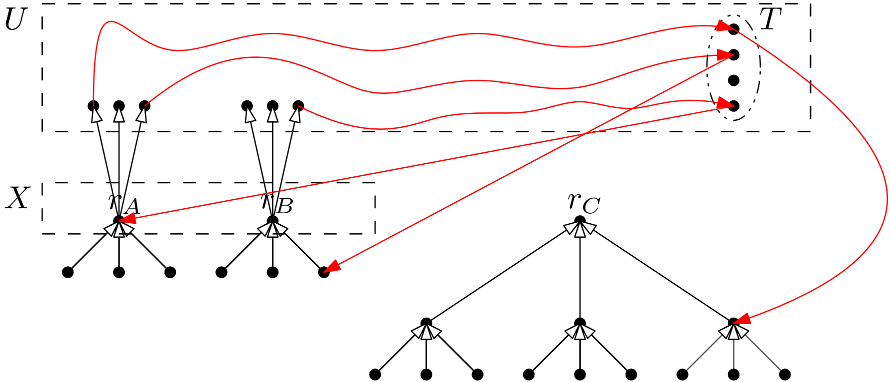

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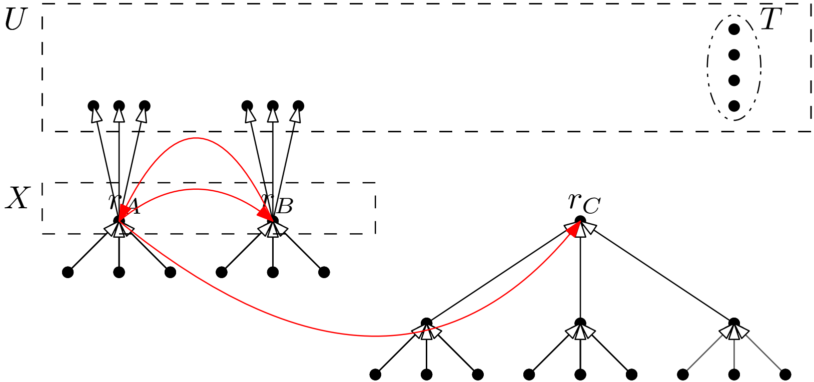

Hairy in-arborescences

Theorem [Klimošová and M., two weeks ago]

Let T be a **hairy** in-arborescence. There exists an integer $g(T)$ such that every digraph D with $\delta^+(D) \geq g(T)$ contains a subdivision of T

Final remarks

We aim to prove this...

Conjecture [Mader, 1985]

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Let T be an oriented tree. There exists an integer $f(T)$ such that every digraph D with $\delta^+(D) \geq f(T)$ contains a subdivision of T

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Our proof relies on the "regularity" of in-arborescences

Other parameters: chromatic number

$\chi(D)$ is simply the chromatic number of its underlying graph

Theorem [Burr, 1980]

For every oriented forest T with n vertices, there exists an integer $f(T)$ ($\approx (n-1)^2$) such that every digraph D with $\chi(D) \geq f(T)$ contains T

Conjecture [Burr, 1980]

Every digraph with chromatic number $2n-2$ contains every oriented tree of order n as a subdigraph

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Every digraph D such that $\vec{\chi}(D) \geq 4^{n^2-2n+1}(n-1)+1$ contains a subdivision of $\vec{K}_n$

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How to get better bounds?

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How does f look like?

For all positive integers k and ℓ such that $\ell \geq 2$,

- $f(1, \ell) = \ell$
- $f(k, \ell) = t(k, \ell) \cdot (\ell - 1) \cdot k + t(k, \ell)$,

  where $t(k, \ell) := f(k-1,\ b(k-1, \ell) \cdot (\ell - 1) + 1) \cdot b(k-1, \ell)$.