ON COUNTING INTERVAL SIZES OF INTERVAL GRAPHS

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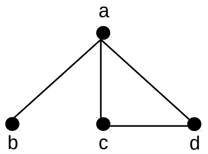
Purpose

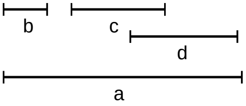
- The problem of counting interval sizes of an interval graph
- Open Problems
- Conjectures

Contents

Interval Graph and Interval Model

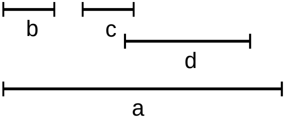
- Interval graph G: Intersection graph of intervals $\mathcal{I}$ in a real line
- Interval model M: Collection of intervals $\mathcal{I}$





Interval Order

- Partial order (order): $P(X, \prec)$
- Interval order: X = set of intervals I_i, s.t. $I_i \prec I_j$ iff I_i lies entirely at the left of I_j



Our World

INTERVAL GRAPHS

INTERVAL ORDERS

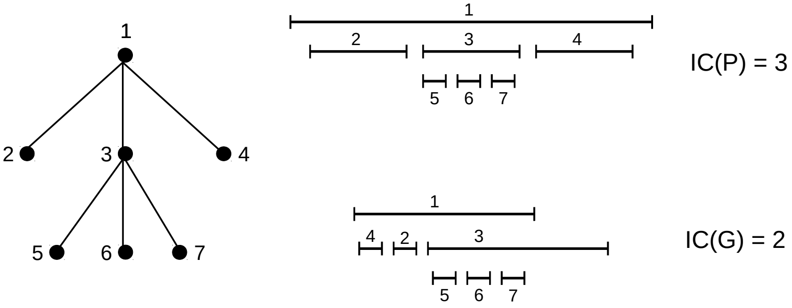
Interval Count

For a model M of an interval graph G,
interval count of M,
$IC(M) =$ number of distinct interval sizes in M

$IC(G) = \{min_{IC(M)} | IC(M)$ is a model of $G\}$

Similarly, define
interval count, $IC(P)$ of an order P

Example



The Problem

Given a graph G (an order P, determime $IC(G)$ ($IC(P)$)

First introduced by Ronald Graham, in late 70's

A quotation about the problem

> *The classes k-LengthINT [graphs having interval count k] were introduced by Graham as a natural hierarchy between unit interval graphs and interval graphs (. . .). Even after decades of research, the only results known are curiosities that illustrate the incredibly complex structure of such a class, very different from the case of unit interval graphs.*

by Pavel Klavik
Extension Properties of Graphs and Structures
Ph.D. Thesis, Charles University, 2017 (page 365)

$IC(G) = 1$

Theorem:
(i) $IC(G) = 1$ iff G is a unit interval graph
(ii) G is a unit interval graphs iff it does not contain induced claws

Similarly for orders

Complexity of finding the IC parameter

Given a graph G and an integer $k > 0$, decide if $IC(G) \leq k$

Polynomial time (linear) if $k = 1$.
Open for any fixed $k \geq 2$.

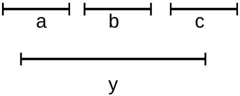
A Lower Bound

NESTED INTERVALS

Let $P = (X, \prec)$ be an order.
Define
nested relation: $\subset_A (P)$

$b \subset_A y$ iff $\exists a, b, c, y \in X$ s.t.
$a \prec b \prec c$ and $y || \{a, x, b\}$

b is forced to be properly included in y



Then: $IC(P) > 1$

k-nested intervals

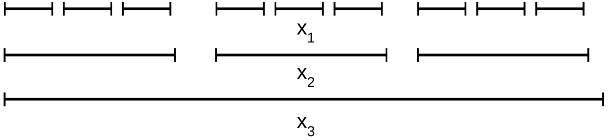
$P(X, \prec)$ s.t.

$x_1 \subset_A x_2 \subset_A x_3 \ldots \subset_A x_{k+1}$

The intervals $x_1, \ldots, x_{k+1}$ are k-*nested*

The *nested height* of $P(X, \prec)$:

$max_k | P$ contains a sequence of $k-$nested intervals



Lower Bound

For and order $(P, \prec)$,

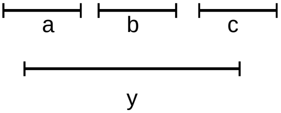
$IC(P) \geq$ nested height(P)

The nested height can be found in polynomial time:
Cerioli, Oliveira and Szwarcfiter (2011)

Linear time:
Klavik (2017)

$IC(G) = 1$

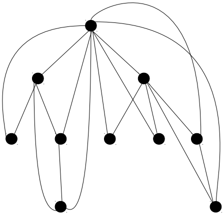
G is claw-free *iff* $IC(G) = 1$



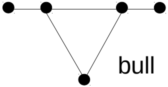
Increasing interval lengths

- Caterpillars, threshold graphs $\rightarrow IC(G) \leq 2$
(Leibowitz 1978)
- Starlike-threshold $\rightarrow IC(G) \leq 2$
(Cerioli and Szwarcfiter (2006)
- Generalized threshold graphs $\rightarrow IC(G) \leq 2$
(Cerioli, Oliveira and Szwarcfiter 2014)
- Trivially perfect graphs $\rightarrow IC(G) \leq k$
(Cerioli, Oliveira and Szwarcfiter 2011)
- Extended bull-free graphs $\rightarrow IC(G) \leq k$
(Cerioli, Oliveira and Szwarcfiter 2011)

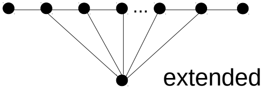
Increasing interval lengths



Trivially Perfect



bull



extended bull

NON-Intuition

Let $P = (X, \prec)$ be an interval order, $IC(P) = 2$
Wlog, let 1 be the least interval size in a model for P. Define

$$\theta(P) = \{\alpha \in \Re | \alpha > 1 \wedge \exists \text{ model } \{1, \alpha\} \text{ for } P\}$$

Example:
P is the order satisfying graph $K_{1,t+2}$, $t \geq 1$

$$\Rightarrow \theta(P) = (t, \infty)$$

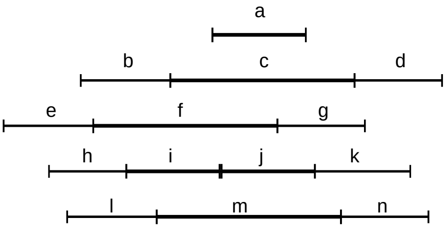
Question:
Is $\theta(P)$ always unbounded ?

NON-Intuition

Conjecture: $\theta(P)$ is unbounded.

Answer: Not true !

Theorem (Fishburn 1984):
Let P be the order as below. Then $\theta(P) = (1, 2)$

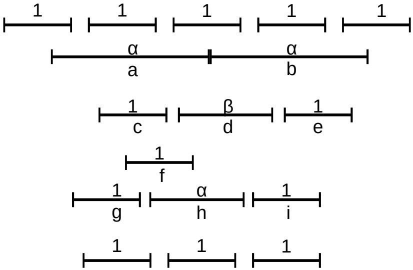


NON Intuition

INTUITION: $\theta(P)$ is continuous !
ANSWER: No !

Theorem (Fishburn 1984)
For each integer $k \geq 2$, there exists an order $P = (X, \prec)$ with $IC(P) = 2$, such that

$$\theta(P) = (2 - 1/k, 2) \cup (k, \infty)$$



NON Intuition

Moreover, the number of disjoint intervals can be arbitrary !

Theorem (Fishburn 1984)
For each integer $k \geq 2$, there exists an order $P = (X, \prec)$ with $IC(P) = 2$, such that

$$\theta(P) = (k, 2k-1) \cup (2k-1, \infty)$$

Corollary: For each integer $m \geq 2$, there is an order $P = (x, \prec)$ with $IC(P) = 2$, such that $\theta(P)$ is the union of m disjoint open intervals

NON-INTUITION

REMOVAL OF A VERTEX

CONJECTURE (Graham): For a graph G and a vertex $v \in V(G)$, if $IC(G) = k + 1$ then $IC(G - v) = k + 1$ or k.

Answer:
True, if $k = 2$, and false otherwise (Leibowicz 1978)

Leibowicz described examples of a graph whose interval count decreases by 2, by removing one vertex !

Conjecture (Trotter): $IC(G)$ can decrease arbitrarily, by removing one vertex.

ANSWER: OPEN

Positive results

RESTRICTED: Two sizes with a given size partition

Given a graph G and a bipartition $V_1 \cup V_2 = V(G)$ decide if G admits a model with two interval sizes, such that, the intervals of V_1 have the same size, and the same applies to V_2.

Can be solved in polynomial time
(Joos, Lowesnstein, Oliveira, Rautenbach, Szwarcfiter (2014).

However: It employs linear programming

Questions:
1) Is there a "purely" combinatorial algorithm ?
2) Can the method be extended, say, for a 3-partition ?

Positive results

$\{0,1\}$-Models

Theorem (Skrien 1984): Let S be the set of simplicial vertices of a graph G. Then G admits a model using only interval sizes 0 and 1 iff there are orientations O_1 of $G \setminus S$ and O_2 of $\overline{G}$, such that

- $O_1 \cup O_2$ is a transitive orientation
- $O_1 O_2 \cup O_2 O_1 \cup O_2 O_2 \subset O_2$,
  where AB denotes the set of pairs ab, such that $ax \in A$ and $xb \in B$

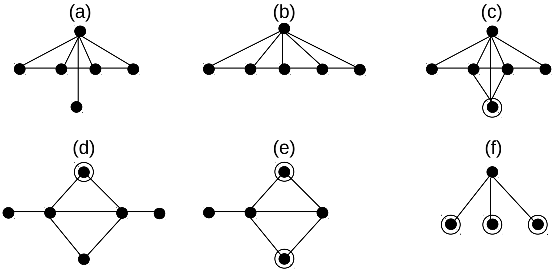
It leads to an algorithm for constructing $\{0,1\}$ models, if existing.
Complexity: $O(n^3)$

Positive results

$\{0,1\}$-Models

Forbidden subgraph chracterization:

Theorem (Rautenbach and Szwarcfiter 2012):
G admits a $\{0,1\}$-Model iff G does not contain any of the following induced subgraphs:



Positive results

$\{0,1\}$-Models for orders

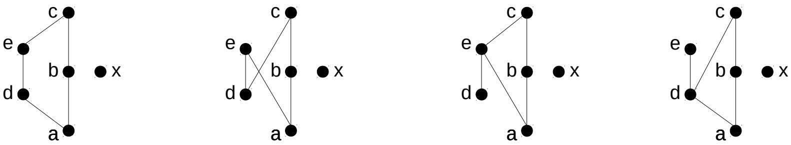
Forbidden suborder characterization:

A vertex is *co-simplicial* if it is simplicial in the co-comparability graph of the order.

Theorem (Boyadzhiyska, Isaak, Trenk 2017): The following are equivalent for an order P:

- P admits a $\{0,1\}$-model.
- In each $\mathbf{3}+\mathbf{1}$ suborder of P, the middle element of the chain is co-simplicial.
- P does not contain any of the following induced suborders.

Forbidden Suborders



Positive results

$(1,2)$-Models for orders

Boyadzhiyska, Isaak, Trenk (2017)
Additionally described a characterization for the orders admitting an $(1,2)$-model.
Moreover, a polynomial time construction of the corresponding models.

The proof and construction employ an accordingly defined weighted order.

Negative results

Pe'er and Shamir (1997) proved the following hardness result, related to the interval count problem:

Given an interval graph G and a pre-described length $\in N$ for each vertex, it is NP-complete to decide if G admits a representation such that the interval corresponding to each vertex has its pre-described length.

Containment relations of two length models of graphs

Let $a, b \in \Re, a < b$

Denote $\text{LEN}(a, b)$ = class of graphs admitting an (a, b)-model.

NATURAL QUESTION:
Given classes $\text{LEN}(a, b)$ and $\text{LEN}(a', b')$,
is $\text{LEN}(a, b) \subseteq \text{LEN}(a', b')$?

A simple containment case

Lemma (Scaling Lemma):

Let $a', a \neq 0$ and $\frac{b}{a} = \frac{b'}{a'}$

Then $\mathrm{LEN}(a', b') = \mathrm{LEN}(a, b)$.

Proof: To transform an (a', b')-model into an (a, b)-model:
multiply the model by $\frac{a}{a'}$
and
To transform an (a, b)-model into an (a', b'):
multiply by $\frac{a'}{a}$

NON-INTUITION ?

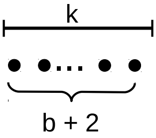
What about the possible inclusions between LEN(a', b') and LEN(a, b),
for $a < b$ and $a' < b'$

We have proved that there are no containments between any pairs of such classes of graphs,
except the following simple special case:

Lemma: $G \in$ LEN(a, b) *iff* $G \in$ LEN(ka, kb),
for all $k > 0$.

LEN$(0, k)$ and LEN(a, b)

Theorem: LEN$(0, k) \not\subseteq$ LEN(a, b)
and LEN$(a, b) \not\subseteq$ LEN$(0, k)$
for all $k, a > 0$



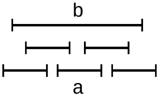
Proof: First, let $a \geq 1$.

Let $G \in$ LEN$(0, k)$, be a graph whose model contains the above model, as a submodel.

LEN$(0, k)$ and LEN(a, b)

The model contains $b + 2$ intervals of size 0,
and an interval of size k, universal to those of size 0.
At least b of those $b + 2$ intervals are nested to the universal interval
Suppose there is an (a, b)-model for this graph
The nested intervals have size $a > 0$, and the universal interval size b
Since there are b nested intervals of size a each,
the universal interval must have size $> ab$, a contradiction,
That is, $G \notin$ LEN(a, b)

LEN$(0, k)$ and LEN(a, b)



Conversely,
Let $G \in$ LEN(a, b), be a graph whose model contains the above as a submodel
The model is basically unique, up to reflexions.
G contains an induced P_5, and a vertex w universal to the P_5.
The middle vertex v of the P_5 cannot be represented by a 0-length interval, because it is not simplicial.
Therefore, in a LEN$(0, k)$-model of G, v has length b.
But v is nested to w
Therefore w must have length $> b$, a contradiction.
Hence, $G \notin$ LEN$(0, k)$

$\mathrm{LEN}(0, k)$ and $\mathrm{LEN}(a, b)$

Let $0 < a < 1$

By the Scaling Lemma and using the first part of the proof,
$\mathrm{LEN}(0, k) \not\subseteq \mathrm{LEN}(1, b/a)$ and $\mathrm{LEN}(1, b/a) \not\subseteq \mathrm{LEN}(0, k)$,
Again by the Scaling Lemma, $\mathrm{LEN}(a, b) = \mathrm{LEN}(1, b/a)$
The result follows

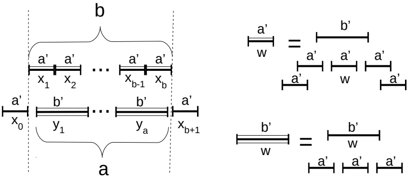
LEN(a', b') and LEN(a, b), $a, b \neq 0$

Theorem: LEN$(a', b') \not\subseteq$ LEN(a, b),
for all rational $0 < a' < b'$, $0 < a < b$, such that $\frac{b'}{a'} \neq \frac{b}{a}$

First: assume a, a', b, b' natural numbers, and $\frac{b'}{a'} < \frac{b}{a}$.
Outline of the proof LEN$(a', b') \not\subseteq$ LEN(a, b):

Build an (a', b')-model M', as a function of a', b', a, b.
Let G be the graph associated to M', then $G \in$ LEN(a', b').
Next, prove that $G \notin$ LEN(a, b).
In the scheme, there are single bar intervals, double and triple bars intervals. The single bar ones correspond just to regular intervals in an interval model of a graph. The double and triple bar intervals are in fact *hyperintervals*, and correspond to sets of special intervals. These sets are depicted in the figure. This completes the description of the model.

LEN(a', b') and LEN(a, b), $a, b \neq 0$



(i) (ii)

Model M' satisfies the following constraints:

- $r(x_i) - \ell(x_i) = a'$, for all $0 \leq i \leq b+1$
- $r(y_i) - \ell(y_i) = b'$, for all $1 \leq i \leq a$
- $\ell(x_{i+1}) = r(x_i)$, for all $0 \leq i \leq b$
- $\ell(y_1) = r(x_0) + \epsilon$ and $\ell(y_{i+1}) = r(y_i) + \epsilon$, for all $1 \leq i < a$, where $0 < \epsilon < \frac{ba' - ab'}{a}$

From the constraints it follows that $r(y_a) = \ell(x_{b+1})$
Therefore, the y_i-intervals all lie between $\ell(x_1)$ and $r(x_b)$, as suggested by the scheme.

$G \in \mathrm{LEN}(a', b') \setminus \mathrm{LEN}(a, b)$

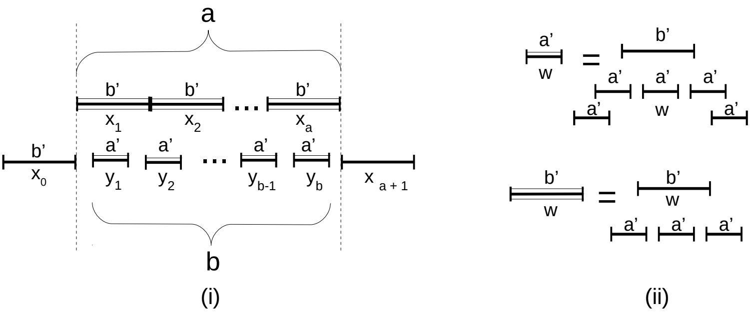
Model M' is an (a', b')-model. Let G be a graph satisfying it. Show that $G \notin \mathrm{LEN}(a, b)$. Suppose the contrary, and let M be an (a, b)-model for G. The following must hold for M:

- y_i is the center of a claw, so it must have length b. for all $1 \leq i \leq a$.
- x_i is adjacent to the center of a P_5, but not to the other vertices of the P_5. So, it must be nested and therefore has length a, for all $1 \leq i \leq b$.
- $r(x_b) - \ell(x_1) \leq ab$ and $r(y_a) - \ell(y_1) > ab$.

From the last condition, it follows:
$r(x_b) - \ell(x_1) < r(y_a) - \ell(y_1)$, implying that the intervals $y_1, \ldots, y_a$ cannot lie between x_0 and x_{b+1}, contradiction. Hence no such model can exist.

The remaining cases

The case $\frac{b'}{a'} > \frac{b}{a}$ is similar.
It requires the use of a different model, shown below:



The result can be extended to arbitrary rational numbers, by using the Scaling Lemma and least common multiples.

OPEN PROBLEMS

PROBLEM 1:
Extend the result of $\mathrm{LEN}(a', b') \not\subseteq \mathrm{LEN}(a, b)$ to possibly allow a', b', a, b to be irrationals.

OPEN PROBLEMS

PROBLEM 2 (Fishburn): For an integer k, determine the least size of the ground set of an order, having interval count at least $k > 1$. That is, let $\sigma(k) = min\{|X| \text{ s.t. } \exists P(X, \prec), IC(P) \geq k\}$

Question:
$\sigma(k) = ?$

Conjecture (Fishburn 1985):
For all $k > 1$
$\sigma(k) = 3k - 2.$

OPEN PROBLEMS

Fishburn proved the conjecture true, in general, for $k \leq 7$, and open for $k > 7$.

Special cases: We proved the conjecture holds for some restricted classes of orders.

OPEN PROBLEMS

PROBLEM 3:
Given a graph G and fixed reals a, b, $0 \leq a < b$,
Does $G \in LEN(a, b)$?

OPEN PROBLEMS

PROBLEM 4:
Given a k-partition of the vertices of an interval graph G, is there a model for G, such that each interval size class corresponds to a part of the partition ?
In particular, what about $k = 3$?

OPEN PROBLEMS

PROBLEM 5: *Arc count*
Given a circular-arc graph G, what is the least number of distinct arc sizes in a circular-arc model of G ?

THANK YOU

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