Open Problems presented at the
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1 Interval Subclasses Recognition

Presented by Jayme Luiz Swarcfiter (jayme@cos.ufrj.br).

Given a graph G, we say that G is an *interval graph* if G is the intersection graph of intervals in $\mathbb{R}$. And given a partial order $P = (X, \prec)$, we say that P is a *interval order* if X ca be related to a set of intervals such that $I \prec I'$ if and only if I lies entirely to the left of I', for every $I, I' \in X$, $I \neq I'$. Given a graph G (partial order P), we denote by $IC(G)$ ($IC(P)$) the minimum number of differently sized intervals that produce G (P). For a positive value $k > 1$, let $\sigma(k)$ be the minimum size of the ground set of a partial order P such that $IC(P) \geq k$. In 1985, Fishburn conjectures the following.

Conjecture 1 *For every $k > 1$, we get that $\sigma(k) = 3k - 2$.*

He proved that it holds for $k \leq 7$, but the conjecture is still open for every value bigger than 7.

Given positive integers a and b, we define $LEN(a, b)$ as the class of graphs that can be represented by intervals of size a and b, where $0 < a < b$.

Question 1 *Can one recognize in polynomial time the graph class $LEN(a, b)$?*

The question above is answered positively for $LEN(0, 1)$ and $LEN(1, 2)$. In addition to these, one can consider the problem where, given a k-partition of G, one has to decide whether there exists an interval model for G such that the partition defined by the different interval sizes coincide with the given partition. Observe that a positive answer implies that $IC(G) \leq k$, but not the contrary. This problem is open even for $k = 3$. Finally, we mention that many of the investigated aspects can be adapted to arc-circular graphs and the related problems on these graphs are largerly open. However, it is expected that these problems will be much harder on arc-circular graphs, given that already the unitary arc-circular graphs differ from the proper arc-circular graphs, which is not the case for intervals.

2 Coloring games

Presented by Nicolas Martins (nicolasam@lia.ufc.br).

Given a graph G and an interger k, a coloring game is usually defined as a game where two players (Alice and Bob) alternate coloring the vertices of G in a proper way using k colors; the objetive of Alice is to be able to finish the coloring, while Bob's objective is to stop Alice from doing so. There are many variations of this kind of problem (for instance, Alice and Bob might be restricted to applying the greedy strategy). Here, we propose a different kind of game, where Bob's turn consists of, instead of coloring a vertex, adding an edge to G. The objectives are the same and many questions can be posed in this context. To begin with, we would like to answer the following questions:

Question 2 *What is the complexity of deciding whether Alice has a winning strategy?*

A natural variation is, given also two other intergers a and b, decide whether there is a winning strategy for Alice when Alice colors a vertices of G, and Bob is allowed to add at most b edges to G each turn.

Question 3 *What is the complexity for some fixed values of k, a and b?*

Up to our knowledge, this is a newly defined type of game, i.e., no previously studied version allows for modifications of G.

3 Dynamic Optimality

Prestend by Victor Campos (campos@lia.ufc.br).

Given a grid G with n columns and m rows, denoted by $G_{m\times n}$, and a subset $S \subseteq V(G)$, we say that S is *arboreally satisfied* if, for every $x, y \in S$, either x and y lie on the same row or column, or there exists $z \in S$ which lies within the subgrid of G having x and y as opposed corner vertices (this is called the *rectangle (x,y)*). If these conditions are not satisfied for a pair x, y, we say that (x, y) is *unsatisfied*.

The offline problem consists of, given a grid $G = G_{m\times n}$, and a subset $S \subseteq V(G)$ containing exactly one vertex of each row, find the smallest $S' \subseteq V(G)$ that is arboreally satisfied and that contains S. We denote the size of such a smallest subset by OPT_S. Here we are interested in the online version of the problem, i.e., the version where at each time stamp, only one row is revealed together with the vertex of S lying on that row.

The *greedy algorithm* applied to the online version works as follows. Given the new row i and the vertex (i, j) of S lying in that row, for each new unsatisfied rectangle. Observe that this must contain vertex (i, j). The following conjectures have been proposed:

Conjecture 2 *There exists an online algorithm that obtains a solution S' such that $|S'| = O(OPT_S)$.*

Conjecture 3 *The greedy algorithm satisfies Conjecture 2.*

The best know result says that the greedy algorithm obtains a solution S' with $|S'| = O(\log n \cdot OPT_S)$ (recall that n is the number of columns). Also, the complexity of the original offline problem is unknown, but it is known to be NP-complete when the restriction of S having exactly one vertex per row is lifted.

An approach to the problem has been to consider only a subset of the rectangles. A rectangle (x, y) with $x = (i, j)$ and $y = (i', j')$ is called *positive* if $i' < i$ and $j < j'$; and it is called *negative* if $i' < i$ and $j < j'$. We say that S is $\boxslash$-*satisfied* if it contains no positive unsatisfied rectangle; similarly, we say that S is $\boxbslash$-*satisfied* if it contains no negative unsatisfied rectangle. One can apply the greedy algorithm in an analogous way to obtain a $\boxslash$-satisfied solution $S' \supseteq S$, as well as a $\boxbslash$-satisfied solution S''; we denote $|S'|, |S''|$ by $OPT_S^{\boxslash}, OPT_S^{\boxbslash}$, respectively. It is known that:

$$OPT_S^{\boxslash} \leq OPT_S; \text{ and}$$

$$OPT_S^{\boxbslash} \leq OPT_S.$$

We propose the following online algorithm. First, we apply the $\boxslash$-greedy and $\boxbslash$ algorithms; then, we run the greedy algorithm to obtain an arboreally satisfied set. We denote the size of the obtained solution by LB_S. We conjecture the following:

Conjecture 4 $LB_S = O(OPT_S)$.

Conjecture 5 $LB = O(\log \log n \cdot OPT_S)$.

Conjecture 6 $LB_S \geq OPT_S$.

Observe that if Conjectures 4 and 6 above hold, then the first two conjectures follow.

4 Decomposing $E(G)$ into paths

Presented by Guilherme Mota (mota@ime.usp.br).

Given a graph G, a *decomposition of* G is a partition of $E(G)$. The path on 5 edges is denoted here by P_5'. We consider the following conjecture.

Conjecture 7 *If G is a 5-regular graph with a perfect matching, then G admits a decomposition into P_5''s.*

It is known that if we add the constraint that G is either C_3-free or C_4-free, then the conjecture holds. We also believe that the conjecture actually holds for $(2d + 1)$-regular graphs with a perfect matching.

In addition to this, we mention the Gallai's Decomposition Conjecture:

Conjecture 8 *Every graph G on n vertices admits a decomposition into at most* $\lfloor \frac{(n+1)}{2} \rfloor$.

The above conjecture has been proved for graphs with maximum degree at most 5 (Bonamy and Perret'18), and for C_3-free planar graphs (Botler, Jimenez and Sambinelli).

5 Connected Greedy Coloring

Presented by Ana Shirley Silva (anasilva@mat.ufc.br).

Given an order $(v_1, \ldots, v_n)$ of the vertices of a graph G, one can color G greedily by giving color 1 to v_1, then iteratively giving the smallest c not appearing in $N(v_i)$, for each $i \in \{2, \ldots, n\}$. A *greedy coloring* is any proper coloring obtained by applying the greedy algorithm. It is not hard to verify that there always exists an order of G that produces a coloring with $\chi(G)$ colors. Therefore, it makes no sense to define the minimization problem, and instead one define the *Grundy number of* G as the maximum integer k for which G has a greedy coloring with k colors; it is denoted by $\Gamma(G)$.

Now, we say that an order $(v_1, \ldots, v_n)$ is *connected* if $G[\{v_1, \ldots, v_i\}]$ is connected for every $i \in \{1, \ldots, n\}$. We then talk about *connected greedy coloring* and *connected Grundy number*, denoted by $\Gamma_c(G)$. However, now it is not true that there always exists a connected order that produces a coloring with $\chi(G)$ colors. Therefore, we define the *connected chromatic number* of G as the minimum $\chi_c(G)$ for which G admits a connected greedy coloring. One interesting aspect is that this is always at most $\chi(G) + 1$ [3].

$$\chi(G) \leq \chi_c(G) \leq \chi(G) + 1.$$

In the same article, they prove that it is NP-hard to decide whether $\chi_c(G) = \chi(G)$, and more recently it has been proved that it remains so even when restricted to C_k-free graphs, for every fixed $k \geq 3$, to line graphs, and to P_k-free graphs, for every fixed $k \geq 9$ [9]. By defining $\chi'_c(G)$ analogously on edges, in [9] we presented a construction of a graph G such that $\chi'_c(G) = \chi'(G) + 1$. However, the constructed graphs all are Class 1 graphs, i.e., they have chromatic index equal to $\Delta(G)$. We then ask whether there exists a graph G with $\chi'(G) = \Delta(G) + 1 < \chi'_c(G)$. In other words:

Question 4 *Does $\chi'_c(G) \leq \Delta(G) + 1$ always hold?*

If the answer for the above question is “no”, then we can also ask:

Question 5 *What is the complexity of deciding whether $\chi'_c(G) \leq \Delta(G) + 1$?*

6 Nash coloring continuity

Presented by Allen Ibiapina (allenr.roossim@gmail.com).

Panagopoulou and Spirakis [10] introduced the following coloring game on a graph G. Each vertex of G is a player that has to choose a color in the set $\{1, \ldots, n\}$; the payoff of $v \in V(G)$ is 0 if v chooses the same color as one of its neighbors, and it is the number of palyer that choose the same color as v (including v), otherwise. A *Nash equilibrium* are states of the game that are susteinable, i.e., such that no player can increase its payoff by changing color. Observe that such a state must be a proper coloring of G as otherwise a vertex with payoff 0 could increase its payoff just by choosing a color not being used. Therefore, one can define a a variation of coloring that models this game. A *Nash coloring of* G is a proper coloring $(S_1, \ldots, S_k)$ such that every $v \in S_i$ has a neighbor in S_j, for every j such that $j \neq i$ and $|S_j| \geq |S_i|$. As observed in [10], there are Nash colorings with $\chi(G)$ colors; hence one define the *Nash number of* G as the maximum k for which G has a Nash coloring with k colors; it is denoted by $Nn(G)$.

More recently, Havet, Sampaio and myself have proved that there are graphs that admit Nash colorings only with $\chi(G)$ and $Nn(G)$ colors. In fact, it is not hard to verify that the graph obtained from $K_{n,n}$ by removing a perfect matching has this property. Therefore, we define the *Nash spectrum of* G as the set $S_N(G)$ of values k for which G admits a Nash coloring with k colors, and we say that a graph G is *Nash continuous* if $S_N(G) = \{\chi(G), \chi(G)+1, \dots, Nn(G)\}$. We ask whether a result similar to one proved in the context of b-colorings [2] also holds for Nash colorings.

Question 6 *Does it hold that there exists a graph G with $S_N(G) = S$ for every non-empty finite $S \subseteq \mathbb{N} - \{1\}$?*

Also, we ask the following:

Question 7 *Can one decide in polynomial time whether a given graph G is Nash continuous?*

7 Backbone coloring

Presented by Júlio Araújo (julio@mat.ufc.br).

Let G be a graph and H a subgraph of G. We say that (G, H) is a *pair*, where H is called *backbone* of G. Given two positive integers q and k, a *q-backbone k-coloring of* (G, H) is a proper k-coloring c of G for which $|c(u) - c(v)| \geq q$ for every $uv \in E(H)$. The *q-backbone chromatic number of* (G, H), denoted by $BBC_q(G, H)$, is the minimum k for which there exists a q-backbone k-coloring of (G, H). Problems regarding backbone colourings were first introduced by Broersma et al. [4], based on colouring problems related to frequency assignment.

It is not hard to see that $BBC_2(G, H)$ is at most $2\chi(G)-1$; it suffices to use colors $1, 3$, up to $2(\chi(G)-1)+1$ in an optimal coloring of G. Also, observe that investigating $BBC_2(G, G)$ amounts to coloring G with $\chi(G)$ colors. Therefore, authors usually investigate what happens when H is restricted to some graph class. In particular, in their seminal paper the authors conjecture that:

Conjecture 9 ([4]) *If G is a planar graph and F is a spanning forest of G, then $BBC_2(G, F) \leq 6$.*

Also, in [5] the authors conjecture:

Conjecture 10 ([5]) *If G is a planar graph and M is a matching of G, then $BBC_2(G, M) \leq 5$.*

There are many partial results for the above conjectures, but it seems we are still far from a definite answer. In particular, there are some partial results about the *circular* version (the first and last colors are considered to be adjacent) of these problems for graphs without cycles of length 4 as subgraphs (e.g. [1]). A good question would be to restrict these questions to this subclass.

8 Matching Cut

Presented by Ignasi Sau (Ignasi.Sau@lirmm.fr).

Consider a connected graph G. An *edge-cut* in a graph G is simply a subset $S \subseteq E(G)$ such that $G - S$ is disconnected. Observe that if (A, B) is a partition of $V(G)$, then the set of edges between A and B is an edge-cut. We represent this cut by $[A, B]$. In this problem we are interested in deciding whether a griven graph has an edge-cut that is also a matching; this is called a *matching cut*. Examples of graphs that do not have matching cuts are the cicle on 3 vertices, and the complete bipartite graph $K_{2,3}$.

It is known that deciding whether a given graph G with maximum degree at most 4 has a matching cut is NP-complete, and that it can be done in polynomial time when $\Delta(G) \leq 3$ (Chvátal, 84). This is done as follows. If G is a tree, then $[v, V(G) - v]$ is a matching cut, where v is any leaf of G. Otherwise, starting with a minimum cycle C, it is always possible to find the desired matching cut.

A natural generalization of this problem is to allow the vertices to be incident to at most d edges of the cut; this is called a *d-cut* (observe that a matching cut is a 1-cut). Recently, Ignasi Sau and Guilheme Gomes (a Ph.D student of Vinicius Santos that has done a 6-months internship in Montpellier) have proved that it is NP-hard to decide whether a $(2d + 2)$-regular graph has a d-cut; and that it can be done in polynomial time for graphs with maximum degree at most $d + 2$. They conjecture that the remaining gap is also polynomial:

Conjecture 11 (Sau and Gomes) *For every $d \geq 1$, one can decide in polynomial time whether a given graph G with maximum degree at most $2d + 1$ has a d-cut.*

Related to this, one can define an *internal partition of G* as a partition of $V(G)$ into (A, B) such that $d_A(x) \geq d_B(x)$ for every $x \in A$, and $d_B(x) \geq d_A(x)$ for every $x \in B$.

Conjecture 12 *For every fixed r, there exists n_r such that every r-regular graph with at least n_r vertices admits an internal partition.*

Note that if the d-CUT problem were NP-hard for $(2d - 1)$-regular graphs, it would imply that Conjecture 12 is false for odd values of r, unless $\mathsf{P} = \mathsf{NP}$. Hence, it is improbable that the degree bound of $2d+2$ of the hardness result could be improved, which supports Conjecture 11.

9 Path Vertex Cover Problem

Presented by Phablo Moura (phablo@ime.usp.br).

Given a graph G, a *path cover of G* is a family $\mathcal{P}$ of vertex disjoint paths in G such that $\bigcup_{P \in \mathcal{P}} V(P)$; the minimum size of a path cover of G is denoted by $\mu(G)$. The following conjecture has been proposed:

Conjecture 13 (Magnant and Martin'09 [8]) *If G is a k-regular graph, then $\mu(G) \leq \frac{n}{k+1}$.*

Magnant and Martin also proved that it holds when $k \leq 5$. The following has been proved by Reed in 96.

Theorem 1 (Reed'96 [6]) *If G is a cubic connected graph, then $\mu(G) \leq \lceil \frac{n}{9} \rceil$.*

Reed also conjectures the following, which has been proved by Yu in 2016 [11].

Conjecture 14 *If G is a cubic 2-connected graph, then $\mu(G) \leq \lceil \frac{n}{10} \rceil$.*

The following theorem has also been proved recently.

Theorem 2 (Han'17 [7]) *For every constant c and α, there exists n_0 such that every $\lceil c \cdot n \rceil$-regular graph G on $n \geq n_0$ vertices admits a packing of vertex-disjoint paths of size at most $\lfloor c^{-1} \rfloor$ that covers all vertices of G except at most $\alpha \cdot n$.*

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