ADDITIVE PROPERTIES OF A PAIR OF SEQUENCES

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Abstract. Motivated by a question of Sárközy, we investigate sufficient conditions for existence of infinite sets of natural numbers A and B such that the number of solutions of the equation $a+b=n$ where $a \in A$ and $b \in B$ is monotone increasing for $n > n_0$. We also examine a generalized notion of Sidon sets. That is, sets A, B with the property that, for every $n \geq 0$, the equation above has at most one solution, i.e., all pairwise sums are distinct.

1. Introduction

For a given set $A \subset \mathbb{N}_0$ of non-negative integers, here and throughout the paper, the *counting function* $A(n)$ is defined as the number of elements of A not exceeding n, i.e., $A(n) = |A \cap \{0,1,2,\dots,n\}|$. Consider the following functions

$$\begin{aligned} r(A,n) =& |\{(a_1,a_2) \in A \times A : a_1 + a_2 = n\}|, \\ r_1(A,n) =& |\{(a_1,a_2) \in A \times A : a_1 + a_2 = n \text{ and } a_1 \leq a_2\}|, \\ r_2(A,n) =& |\{(a_1,a_2) \in A \times A : a_1 + a_2 = n \text{ and } a_1 < a_2\}|. \end{aligned}$$

A well-studied problem concerning these functions is to determine necessary and sufficient conditions on A for their (eventual) monotonicity. Here and throughout the paper, monotonicity refers to monotonicity in n. In other words, for what sets A we can find an n_0 such that $r(A,n+1) \geq r(A,n)$ for all $n > n_0$? Although the three functions look similar, and in fact $|r(A,n) - 2r_2(A,n)| \leq 1$ and $|r_1(A,n) - r_2(A,n)| \leq 1$, the (partial) answers to these questions may be quite different.

Erdős, Sárközy and Sós [3] proved that $r(A,n)$ is eventually monotone increasing if and only if A contains all the positive integers from a certain point on. On the other hand, they obtained only a partial answer for r_1 and r_2. In particular, they proved that if

$$\lim_{n \to +\infty} \frac{n - A(n)}{\log n} = +\infty$$

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then $r_1(A, n)$ is not eventually monotone increasing. (This result was also obtained independently by Balasubramanian [1].)

Also, for $r_2(A, n)$ they proved that if

$$A(n) = o\left(\frac{n}{\log n}\right)$$

then $r_2(A, n)$ cannot be monotone increasing from a certain point on.

Motivated by these results, Sárközy asked the following question in his valuable paper on unsolved problems in number theory [8] (see Problem 4 in [8]).

Problem 1. If A, B are given infinite sets of non-negative integers, what can one say about the monotonicity of the number of solutions of the equation

$$a + b = n,\ a \in A,\ b \in B?$$

We can naturally rephrase this question by defining the following function.

Definition 2. The *representation function* for two sets $A, B \subset \mathbb{N}_0$ is

$$r(A, B, n) = |\{(a, b) \in A \times B : a + b = n\}|.$$

The main goal of the present paper is to give some sufficient conditions on A, B for the monotonicity of this function. This new representation function acts surprisingly different from the prequel. Our main result is as follows.

Theorem 3. *For all* $0 \le \alpha, \beta < 1$, $1/2 < c_1, c_2 \le 1$, *there exist sets* $A, B \subset \mathbb{N}_0$ *such that* $r(A, B, n)$ *is monotone increasing in* n*;*

$$\limsup_{n\to\infty} \frac{A(n)}{n^{c_1}} = \alpha; \qquad \limsup_{n\to\infty} \frac{B(n)}{n^{c_2}} = \beta.$$

In the next sections we develop tools to approach Theorem 3 and prove some related results. Then we will return to the proof of Theorem 3.

2. co-Sidon Sets

Before proving Theorem 3, we introduce a generalized notion of Sidon sets and study some of its properties. Recall that a set $A \subset \mathbb{N}_0$ is called *Sidon* if $r_1(A, n) \le 1$ for all $n \in \mathbb{N}$, i.e., the sums of unordered pairs of elements of A are all distinct. We remark that it is possible to extend the notion of a Sidon set to a pair of sets in different ways. In this paper, we consider the following generalization.

Definition 4. Two sets $A, B \subset \mathbb{N}_0$ are called *co-Sidon* if $r(A, B, n) \le 1$ for all $n \in \mathbb{N}_0$, i.e., the sums $a + b$ are distinct for all $(a, b) \in A \times B$.

Note that if A, B are co-Sidon then $|A \cap B| \leq 1$.

For sets A and B of integers we denote their *sum set* by $A+B = \{a+b : a \in A, b \in B\}$. For simplicity if the set B is a single element b we denote their sum set by $A + b = A + B$.

When A, B are finite sets, we prove a simple but sharp result about $|A|, |B|$.

Proposition 5. *If $A, B \subset \{0, 1, 2, \dots, n\}$ are co-Sidon, then*

$$\min\{|A|, |B|\} \leq \lfloor \sqrt{2n} \rfloor.$$

Furthermore, equality can be obtained for infinitely many values of n.

Proof. Since A and B are finite (and co-Sidon) we have $|A + B| = |A||B|$. Without loss of generality assume $|A| \leq |B|$. Then, $|A|^2 \leq |A + B|$.

Clearly for an element $c \in A + B$ we have $0 \leq c \leq 2n$. However, either 0 or $2n$ is not an element of $A + B$, otherwise we would have $0, n \in A \cap B$ and there would be two distinct solutions to $a + b = n$ with $a \in A$ and $b \in B$. Thus, $|A + B| \leq 2n$ which yields $|A| \leq \lfloor \sqrt{2n} \rfloor$ and the upper-bound is established.

To see that the upper bound is best possible for infinitely many n, consider the following construction for A and B. Let $m \in \mathbb{N}$ be fixed and define

$$A := \{0, m, 2m, \dots, (2m-1)m\}$$

and

$$B := \{0, 1, 2, \dots, m-1, 2m^2, 2m^2+1, 2m^2+2, \dots, 2m^2+m-1\}.$$

Note that $|A| = |B| = 2m$ and $A + B = \{0, 1, \dots, 4m^2 - 1\}$. Therefore A and B are co-Sidon. As $A, B \subseteq \{0, 1, 2, \dots, 2m^2 + m - 1\}$, we can take $n = 2m^2 + m - 1$. This gives

$$2m = \sqrt{4m^2} \leq \sqrt{4m^2 + 2m - 2} = \sqrt{2n} < \sqrt{4m^2 + 4m + 1} = 2m + 1.$$

Hence $\min\{|A|, |B|\} = 2m = \lfloor \sqrt{2n} \rfloor$. As the choice of m was arbitrary, there are infinitely many n for which we can reach the upper bound in the statement of the theorem. $\square$

It is worth to compare the above result to the following theorem of Erdős and Turán [4] on finite Sidon sets.

Theorem 6. *There is an absolute positive constant c such that if $n \in \mathbb{N}$ and $A \subset \{1, 2, \dots, n\}$ is a Sidon set, then $|A| < n^{1/2} + cn^{1/4}$.*

On the other hand, the best known constructions give Sidon sets of size $n^{1/2}$ for infinitely many n (see e.g. [5, 7] for details). The reduction of this gap is a well-known hard problem.

We consider now the case where A, B are infinite co-Sidon. Defining $A_n = A \cap \{0, 1, \ldots, n\}$ and $B_n = B \cap \{0, 1, \ldots, n\}$, we have that A_n,B_n are co-Sidon. So, by Theorem 5, for any n we have

$$\min\{A(n), B(n)\}/\sqrt{n} = \min\{|A_n|, |B_n|\}/\sqrt{n} \le \lfloor\sqrt{2n}\rfloor/\sqrt{n} \le \sqrt{2}.$$

A simple example shows that we can come close to achieving this bound.

Construction 7. Let A be the set of integers which can be written in the form $\sum_{i=0}^{k} \alpha_i 2^{2i}$ where $\alpha_i \in \{0, 1\}$ and $k \in \mathbb{N}$. Let B be the set of integers which can be written in the form $\sum_{i=0}^{k} \alpha_i 2^{2i+1}$ where $\alpha_i \in \{0, 1\}$ and $k \in \mathbb{N}$. It is clear that A and B are co-Sidon and $A + B = \mathbb{N}_0$. It can easily be verified that

$$\liminf_{n\to\infty} \frac{A(n)}{\sqrt{n}} = 1$$
$$\liminf_{n\to\infty} \frac{B(n)}{\sqrt{n}} = \frac{\sqrt{2}}{2}$$
$$\limsup_{n\to\infty} \frac{A(n)}{\sqrt{n}} = \sqrt{3}$$
$$\limsup_{n\to\infty} \frac{B(n)}{\sqrt{n}} = \frac{\sqrt{6}}{2}$$

Thus,

$$\liminf_{n\to\infty} \frac{\min\{A(n), B(n)\}}{\sqrt{n}} = \sqrt{2}/2.$$

Comparing this with the following result of Erdős (see [9, 5]), we conclude that infinite Sidon sets and infinite co-Sidon sets also behave differently. In general, we have more freedom when working with co-Sidon sets.

Theorem 8. *There is an absolute, positive constant c such that for any infinite Sidon set $A \subset \mathbb{N}$ we have*

$$\liminf_{n\to\infty} \frac{A(n)}{\sqrt{n/\log n}} < c.$$

It is also worth mentioning the following theorem of Krückeberg [6] for infinite Sidon sets.

Theorem 9. *There is a Sidon set $A \subset \mathbb{N}$ such that*

$$\limsup_{n\to\infty} \frac{A(n)}{\sqrt{n}} \ge \sqrt{2}/2.$$

The following definition will be useful for us.

Definition 10. We call sets $A, B \subset \mathbb{N}_0$ *perfect* if the sum set $A+B$ is an interval (possibly unbounded) of consecutive integers.

The next proposition will be helpful in building new perfect co-Sidon sets from other co-Sidon sets.

Proposition 11. *Let $A, B \subset \mathbb{N}_0$ be finite perfect co-Sidon sets. Let $c = \max(A) + \max(B) - \min(A) - \min(B) + 1$. Then for any $k \in \mathbb{N}_0$, the sets A and $C = \bigcup_{i=0}^{k}(B+ic)$ are perfect co-Sidon.*

Proof. Let $r = \min(A) + \min(B)$. By assumption, $A+B = \{r, r+1, \ldots, c+r-1\}$. For each i, the sets A and $B+ic$ are co-Sidon. Furthermore, the sets

$$\begin{aligned} A + (B+c) &= \{c+r, c+r+1, \ldots, 2c+r-1\} \\ A + (B+2c) &= \{2c+r, 2c+r+2, \ldots, 3c+r-1\} \\ &\vdots \\ A + (B+kc) &= \{kc+r, kc+r+1, \ldots, (k+1)c+r-1\} \end{aligned}$$

are all pairwise disjoint consecutive intervals. Therefore A and $\bigcup_{i=0}^{k}(B+ic)$ are perfect co-Sidon with sum set $\{r, r+1, \ldots, (k+1)c+r-1\}$. ☐

Clearly the proposition also holds for $C = \bigcup_{i=0}^{\infty}(B+ic)$.

Next we characterize all infinite perfect co-Sidon sets $A, B \subset \mathbb{N}_0$ using the mixed radix representation. Note that both the co-Sidon and perfect properties are invariant under translation of each of the sets (i.e. addition or subtraction by a constant), so without loss of generality we may assume $0 \in A \cap B$.

Theorem 12. *Let $A, B \subset \mathbb{N}_0$ be infinite, such that $0 \in A \cap B$. Then A, B are perfect co-Sidon if and only if there exists an infinite sequence of integers $(k_i)_{i=1}^{\infty}$ such that $\forall i, k_i \geq 2$ and (up to an exchange of A and B),*

$$A = \left\{ \sum_{i=1}^{\infty} k_1 k_2 \ldots k_{2i-2} a_{2i-1} : \forall j, 0 \leq a_{2j-1} < k_{2j-1}, \text{ finitely many } a_{2i-1} \text{ non-zero} \right\}$$

and

$$B = \left\{ \sum_{i=1}^{\infty} k_1 k_2 \ldots k_{2i-1} a_{2i} : \forall j, 0 \leq a_{2j} < k_{2j}, \text{ finitely many } a_{2i} \text{ non-zero} \right\}.$$

Proof. A sum of the form $\sum_{i=1}^{\infty} k_1 k_2 \ldots k_{i-1} a_i$ where $0 \leq a_j < k_j$, and only finitely many a_i are non-zero, is precisely the so-called *mixed-radix representation* with bases $(k_1, k_2, \ldots, k_i, \ldots)$. Thus the base r representation is the special case where $k_i = r$ for all i. For any sequence $(k_i)_{i=1}^{\infty}$ of integers with $k_i \geq 2$, every non-negative integer is uniquely representable with bases (k_i).

Let $(k_i)_{i=1}^{\infty}$ be a sequence of integers such that $\forall i, k_i \geq 2$. Suppose A and B are of the form determined by the bases k_i as above. As every non-negative integer is uniquely representable by with bases (k_i), A and B are co-Sidon. Also observe that

$$A+B = \left\{ \sum_{i=1}^{\infty} k_1 k_2 \dots k_{i-1} a_i : \forall j, 0 \leq a_j < k_j, \text{ finitely many } a_i \text{ non-zero} \right\}.$$

Thus $A + B = \mathbb{N}_0$ and therefore A and B are perfect.

Now assume that A, B are perfect co-Sidon. Unless $A = B = \{0\}$, we can assume without loss of generality that $1 \in A$. To show that A, B are of the required form, we need to construct a sequence of base elements $(k_i)_{i \in \mathbb{N}}$ that represents A and B as in the statement of the theorem.

Our construction of the integers k_i is recursive. Let $k_0 = 1$. For $t \geq 1$ define $c_t = k_{t-1} k_{t-2} \cdots k_0$ and let

$$k_t = \begin{cases} \max \{a : \{c_t, 2c_t, \dots, (a-1)c_t\} \subset A\}, & \text{if } t \text{ is odd} \\ \max \{b : \{c_t, 2c_t, \dots, (b-1)c_t\} \subset B\}, & \text{if } t \text{ is even} \end{cases}$$

Note that $\forall t > 0, k_t < \infty$. Otherwise, one of A or B contains an infinite arithmetic progression, whose consecutive terms differ by c_t. But as they are co-Sidon, this implies that the other set is finite (in fact of cardinality at most c_t), a contradiction.

Now define two families of sets. Let $A_0 = B_0 = \{0\}$ and for each $t \geq 1$,

$$A_t = \left\{ \sum_{i=1}^{t} k_1 k_2 \dots k_{i-1} a_i : \forall j, 0 \leq a_j < k_j \text{ and } a_{2j} = 0 \right\}$$

and

$$B_t = \left\{ \sum_{i=1}^{t} k_1 k_2 \dots k_{i-1} b_i : \forall j, 0 \leq b_j < k_j \text{ and } b_{2j-1} = 0 \right\}.$$

Note that for all j, $A_{2j} = A_{2j-1}$ and $B_{2j-1} = B_{2j-2}$. Let $A^* = \bigcup_{i=0}^{\infty} A_t$ and $B^* = \bigcup_{i=0}^{\infty} B_t$. It only remains to prove that $A = A^*$ and $B = B^*$. We will use the following claim.

Claim 13. *For all $t \geq 0$*

$$\begin{aligned} A \cap \{0, 1, \dots, k_1 \cdots k_t - 1\} &= A_t \\ B \cap \{0, 1, \dots, k_1 \cdots k_t - 1\} &= B_t. \end{aligned}$$

Proof. Suppose not and let t be minimal such that the claim does not hold. Thus there must exist an $x \in \mathbb{N}$ such that either

$$x \in (A \cap \{0, 1, \dots, k_1 k_2 \cdots k_t - 1\}) \Delta A_t$$

or

$$x \in (B \cap \{0, 1, \ldots, k_1 k_2 \cdots k_t - 1\}) \Delta B_t$$

where Δ denotes the symmetric difference of sets. Pick a minimal such x. Let us assume that t is odd and $t \geq 3$; the proof is trivial for $t = 0$ or $t = 1$ and similar when $t \geq 2$ is even. As t is odd (and minimal) $B_t = B_{t-1} = B \cap \{0, 1, \ldots, k_1 \cdots k_{t-1} - 1\} \subset B \cap \{0, 1, \ldots, k_1 \cdots k_t - 1\}$, thus $B_t \backslash (B \cap \{0, 1, \ldots, k_1 \cdots k_t - 1\})$ is empty.

Now write

$$x = \sum_{i=1}^{t} k_1 k_2 \ldots k_{i-1} a_i$$

in the mixed-radix representation with bases $(k_i)_{i=1}^{\infty}$. Set

$$z = \sum_{i=0}^{\lfloor \frac{t}{2} \rfloor} k_1 \cdots k_{2i} a_{2i+1}$$

and

$$w = \sum_{i=1}^{\lfloor \frac{t}{2} \rfloor} k_1 \cdots k_{2i-1} a_{2i}.$$

By definition, $z \in A_t$, $w \in B_t = B_{t-1}$ and $x = z + w$. By the minimality of t, $B_{t-1} \subset B$, thus $w \in B$. We now distinguish the remaining three cases.

(i) Suppose $x \in (A \cap \{0, 1, \ldots k_1 \cdots k_t - 1\}) \backslash A_t$. Since $x \notin A_t$, we have $x \neq z$, thus $z \in A$ by minimality of x. Now we have that $x, z \in A$ and $0, w \in B$. But $x + 0 = z + w$, contradicting the fact that A and B are co-Sidon.

(ii) Suppose $x \in A_t \backslash (A \cap \{0, 1, \ldots, k_1 \cdots k_t - 1\})$. As $A + B = \mathbb{N}_0$, we can write $x = a + b$ with $a \in A$, $b \in B$. Note that $x \leq k_1 k_2 \cdots k_t - 1$ and this implies $x \notin A$. In particular, $x \neq a$. We claim that $x = b$. If not, then $0 < a, b < x$ and the minimality of x implies that $a \in A_t$ and $b \in B_t$. But $a + b = x \in A_t$, which contradicts the definition of A_t and B_t. Thus we may suppose $x = b$, i.e., $x \in A_t \cap B$.

For $0 \leq i \leq \lfloor \frac{t}{2} \rfloor - 1$, define

$$\alpha_{2i+1} = \begin{cases} k_{2i+1} - a_{2i+1} & \text{if } a_{2i+1} > 0 \\ 0 & \text{if } a_{2i+1} = 0 \end{cases}$$

and

$$\beta_{2i+2} = \begin{cases} 0 & \text{if } \alpha_{2i+1} = 0 \\ 1 & \text{if } \alpha_{2i+1} > 0. \end{cases}$$

Let

$$u = (\alpha_{t-1}0\alpha_{t-4}\ldots\alpha_3 - \alpha_1)_{(k_i)} = \sum_{i=0}^{\lfloor\frac{t}{2}\rfloor-1} k_1\cdots k_{2i}\alpha_{2i+1} \in A_{t-2},$$

$$v = (\beta_{t-1}0\beta_{t-3}0\ldots\beta_2 0)_{(k_i)} = \sum_{i=1}^{\lfloor\frac{t}{2}\rfloor} k_1\cdots k_{2i-1}\beta_{2i}.$$

By definition of k_t, $a_t\prod_{i=0}^{t-1}k_i \in A$ and by minimality of t, we have $u \in A$ and $v \in B$. Clearly, $u \neq a_t\prod_{i=0}^{t-1}k_i$. But $u+x = a_t\prod_{i=0}^{t-1}k_i + v$, contradicting the fact that A and B are co-Sidon.

(iii) Suppose $x \in (B \cap \{0,1,\ldots,k_1\cdots k_t - 1\})\backslash B_t$. Clearly $x \notin A$, otherwise $0, x \in A \cap B$ which contradicts A, B being co-Sidon. Also $x \notin A_t$, otherwise $x \in A_t \cap B$ and we can continue as at the end of case (ii). Thus $x \neq z$, this implies $z \in A$ by the minimality of x. Also $w \in B_t$ implies $x \neq w$. Now $0 + x = z + w$, with $0, z \in A$ and $x, w \in B$ contradicting the fact that A and B are co-Sidon. □

To complete the proof of the theorem, we must show $\forall t > 0, k_t \geq 2$. Suppose that $k_{t_0} = 1$. That is, $c_{t_0} = k_1k_2\cdots k_{t_0-1}$ is in neither A nor B. But then as A and B are perfect co-Sidon, there exist $a \in A$ and $b \in B$ such that $a + b = c_{t_0}$. By assumption, $a, b < c_{t_0}$. But clearly $(a,b) \notin A_{t_0} \times B_{t_0}$ as $A_{t_0} + B_{t_0} \subset \{0,1,\ldots,c_{t_0}-1\}$ contradicting Claim 13. □

Theorem 12 allows us to make a useful observation about the structure of perfect co-Sidon sets.

Corollary 14. *If A and B are infinite perfect co-Sidon sets then for all $m \in \mathbb{N}$ there are infinitely many $n \in \mathbb{N}$ such that $\{n, n+1, \ldots, 2n+m\} \cap A = \emptyset$.*

Proof. As the statement remains true when we translate A or B, it suffices to prove it for A and B with $0 \in A \cap B$. There exists an infinite sequence of integers (k_i) $\forall i, k_i \geq 2$ such that A and B are represented by the bases k_i as in Theorem 12. Fix $m \in \mathbb{N}$ and let t be such that $2\prod_{i=0}^{t-1}k_i - 3 \geq m$ and $(k_t - 1)\prod_{i=0}^{t-1}k_i \in A$. Then by Theorem 12 the next element in A is exactly $\prod_{i=0}^{t+1}k_i$. Let $n = (k_t-1)\prod_{i=0}^{t-1}k_i + 1$. Now

$$\begin{aligned}\prod_{i=0}^{t+1}k_i &= k_{t+1}\{(k_t-1)+1\}\prod_{i=0}^{t-1}k_i \\ &\geq 2\left\{n - 1 + \prod_{i=0}^{t-1}k_i\right\} \\ &\geq 2n - 2 + m + 3 = 2n + m + 1.\end{aligned}$$

Thus $\{n, n+1, \ldots, 2n+m\} \cap A = \emptyset$. Since A is infinite, it follows that for every m there are infinitely many such n. □

It is natural to ask whether all co-Sidon sets A, B are subsets of perfect co-Sidon sets A^*, B^*. The answer turns out to be no as the following proposition shows.

Proposition 15. *The sets $A = \{2^k : k \in \mathbb{N}, k \geq 9\}$ and $B = \{3^l : l \in \mathbb{N}, l \geq 9\}$ are co-Sidon and there are no perfect co-Sidon sets A^*, B^* such that $A \subseteq A^*$ and $B \subseteq B^*$.*

Proof. The Diophantine equation $2^k + 3^l = 2^m + 3^n$ with $k < m$ and $l > n$ has only five solutions (see [10]); all have exponents less than 9. This implies that A and B are co-Sidon.

Note that, for all $n \geq 2^9$, A contains numbers between n and $2n$. That is, for all n, $A \cap \{n, n+1, \ldots, 2n\} \neq \emptyset$. However, if A^* and B^* are perfect co-Sidon sets such that $A \subset A^*$ and $B \subset B^*$, then according to Corollary 14 there is an n with $A^* \cap \{n, n+1, \ldots, 2n+m\} = \emptyset$. □

3. Representation Function

We seek to provide sufficient conditions on A and B so that the representation function $r(A, B, n) = |\{(a, b) \in A \times B : a + b = n\}|$ is (eventually) monotone increasing. For $C \subset \mathbb{N}_0$ let us denote its complement $\overline{C} = \mathbb{N}_0 \backslash C$.

It is easy to see that if either A or $\overline{A}$ is finite and either B or $\overline{B}$ is finite then $r(A, B, n)$ is eventually monotone. To see this, if $\overline{A}$ and B are finite, then for all $n > \max(\overline{A}) + \max(B)$ we have that $b \in B$ implies $n - b \in A$ and thus $r(A, B, n) = |B|$. Also, if $\overline{A}$ and $\overline{B}$ are finite, then for all $n > \max(\overline{A}) + \max(\overline{B})$ we have $r(A, B, n) = n + 1 - |\overline{A}| - |\overline{B}|$. Finally, if A and B are both finite then it is obvious that $r(A, B, n)$ is eventually monotone. So the study is non-trivial only in the case when A and $\overline{A}$ are both infinite.

Proposition 16. *Let $A, B \subset \mathbb{N}_0$ be infinite perfect co-Sidon sets such that $A + B = \mathbb{N}_0$. Then, for any $A' \subset A$ and $B' \subset B$, the representation function $r(A + B', B + A', n)$ is monotone increasing.*

Proof. Note that

$$\begin{aligned} r(A + B', B + A', n) &= r\left(\bigcup_{b \in B'} A + b, \bigcup_{a \in A'} B + a, n\right) \\ &= \sum_{a \in A', b \in B'} r(A + b, B + a, n) \end{aligned}$$

The second equality holds because the unions are disjoint.

From $A+B=\mathbb{N}_0$ it follows that $(A+b)+(B+a)=\mathbb{N}_0+a+b$ and thus each summand is

$$r(A+b,B+a,n)=\begin{cases} 0 & \text{if } n<a+b,\\ 1 & \text{if } n\geq a+b.\end{cases}$$

Therefore, the representation function $r(A+B',B+A',n)$ is monotone increasing. □

It follows from Theorem 12 that sets A and B which are infinite perfect co-Sidon exist. Since the subsets in Proposition 16 are arbitrary, we can construct many sets A and B such that $r(A,B,n)$ is monotone increasing. The next theorem allows us to choose sets A and B whose representation function is monotone and increasing and whose counting functions $A(n)$ and $B(n)$ grow at a controlled rate.

Theorem 17. *Let $A,B\subset\mathbb{N}_0$ be infinite perfect co-Sidon such that $A+B=\mathbb{N}_0$. Let $f:\mathbb{N}_0\to\mathbb{R}$ be such that $A(n)\leq f(n)$ and for every $M>0$ there exists n_0 such that for $n>n_0$ we have $f(n)<n+1-MA(n)$. Then there exists a $B'\subseteq B$ such that*

$$(A+B')(n)\leq f(n) \text{ for all } n\in\mathbb{N}_0$$

and

$$(A+B')(n)\geq f(n)-A(n) \text{ for infinitely many } n\in\mathbb{N}_0.$$

Proof. Let A and B be as in the statement and write $B=\{b_0<b_1<\ldots\}$. By assumption, $b_0=0$. Let us construct $B'\subseteq B$ greedily as follows: set $B'_0=\{0\}$ and for $i>0$ let

$$B'_{i+1}=\begin{cases} B'_i\cup\{b_{i+1}\} \text{ if } \big(A+(B'_i\cup\{b_{i+1}\})\big)(n)\leq f_A(n) \text{ for all } n\in\mathbb{N}_0,\\ B'_i \text{ otherwise.}\end{cases}$$

Then let $B'=\bigcup_{i=0}^{\infty}B'_i$. We claim that this B' satisfies the conditions of the theorem. By the construction,

$$(A+B')(n)\leq f(n) \text{ for all } n\in\mathbb{N}_0.$$

To prove that the other inequality holds for infinitely many values of n, we first need to show that $B\setminus B'$ is infinite. Suppose that $B\setminus B'$ is finite, and let $M=|B\setminus B'|$. Since $A+B\setminus B'=\cup_{b\in B\setminus B'}(A+b)$ we have $(A+B\setminus B')(n)\leq MA(n)$ for every n. Now, clearly

$$\bigcup_{b\in B'}(A+b)=\mathbb{N}_0\setminus\left(\bigcup_{b\in B\setminus B'}(A+b)\right).$$

It follows that $\big(A+B'\big)(n) = n+1-\big(A+(B\backslash B')\big)(n) \geq n+1-MA(n)$ for all n. But, for large enough n, we have $n+1-MA(n) > f(n)$. Then, for large enough n we would have $\big(A+B'\big)(n) > f(n)$, which contradicts the construction of B'. Hence $B \setminus B'$ is infinite.

Therefore, for infinitely many i, we have $b_{i+1} \notin B'$. For such an i we have $B'_{i+1} = B'_i$. Therefore, by definition of B'_{i+1}, there exists n_{i+1} such that $(A+B'_i \cup \{b_{i+1}\})(n_{i+1}) > f(n_{i+1})$. Note that $n_{i+1} \geq b_{i+1}$, because for all $n < b_{i+1}$,

$$\big(A+B'_i \cup \{b_{i+1}\}\big)(n) = \big(A+B'_i\big)(n) \leq f_A(n).$$

Therefore there are infinitely many n such that,

$$(A+B')(n) \geq (A+B'_i)(n) \geq f(n) - A(n).$$

□

Our main theorem follows as a corollary of Theorem 17. We restate it here for easy reference:

Theorem 3. *For all $0 \leq \alpha, \beta < 1$, $1/2 < c_1, c_2 \leq 1$, there exist sets $A, B \subset \mathbb{N}_0$ such that $r(A,B,n)$ is monotone increasing in n;*

$$\limsup_{n\to\infty} \frac{A(n)}{n^{c_1}} = \alpha; \qquad \limsup_{n\to\infty} \frac{B(n)}{n^{c_2}} = \beta.$$

Proof. Suppose we are given constants $0 \leq \alpha < 1$ and $1/2 < c_1 \leq 1$ Let A_0, B_0 be perfect co-Sidon sets such that $A_0(n) = \Theta(n^{1/2})$, $B_0(n) = \Theta(n^{1/2})$ (e.g. Construction 7.) Let $f(n) = \alpha n^{c_1} + d$ where d is a constant large enough such that $f(n) \geq A_0(n)$ for all n. Clearly for all $m > 0$ there exists an n_0 such that for $n > n_0$, $f(n) < n+1-mA_0(n)$. By Theorem 17, there is a $B' \subset B_0$ such that $(A_0+B')(n) \leq f(n)$ for all n and $(A_0+B')(n) \geq f(n) - A_0(n)$ for infinitely many n. Set $A = A_0 + B'$. Then

$$\alpha = \lim_{n\to\infty} \frac{f(n)}{n^{c_1}} \geq \limsup_{n\to\infty} \frac{A(n)}{n^{c_1}} \geq \lim_{n\to\infty} \frac{f(n)-A_0(n)}{n^{c_1}} = \alpha.$$

We can construct B in the same manner. By Proposition 16, the representation function $r(A,B,n)$ is monotone increasing. □

By modifying the previous two proofs, we can restate Theorem 3 with either (or both) limit superiors replaced with limit inferiors. The details are left to the interested reader. Theorem 3 gives a strong answer about the densities of sets A and B with monotone representation function $r(A,B,n)$.

When $c_1 = c_2 = 1$ and $\alpha, \beta \in \mathbb{Q}$ we can restate Theorem 3 by replacing the limit superiors with standard limits.

Theorem 18. *For all rational* $0 \leq \alpha, \beta \leq 1$*, there exist sets* $A, B \subset \mathbb{N}_0$ *such that* A *has density* α*,* B *has density* β *and* $r(A, B, n)$ *is monotone increasing in* n*.*

Proof. We construct A and B using mixed radix representation to describe its elements. Write $\alpha = p_1/q_1$ and $\beta = p_2/q_2$ where $p_i, q_i \in \mathbb{N}$. Set $k_1 = q_1$, $k_2 = q_2$ and $k_i = 2$ for all $i > 2$. Let A_0 be the set of all integers that can be written in the form

$$\sum_{i=0}^{k} k_1 k_2 \cdots k_{2i} a_{2i+1}$$

where for each i, $0 \leq a_{2i+1} < k_{2i+1}$. Similarly let B_0 be the set of all integers that can be written in the form

$$\sum_{i=1}^{k} k_1 k_2 \cdots k_{2i-1} b_{2i}$$

where for each i, $0 \leq b_{2i} < k_{2i}$. Note that A_0 and B_0 are perfect co-Sidon.

Let A' be the subset of A_0 consisting of all those integers whose k_1-digit (in the mixed radix representation) lies in the set $\{0, 1, \ldots, p_1 - 1\}$. As $p_1 \leq q_1$ we must have $p_1 - 1 \leq k_1 - 1$. Thus A' is well-defined. Then $B = A' + B_0$ is the set of all numbers whose k_1-digit lies in $\{0, \ldots, p_1 - 1\}$ that is, B consists of the numbers congruent to $0, 1, \ldots, p_1 - 1 \pmod{q_1}$. The density of this set is clearly p_1/q_1.

Similarly, let B' be the subset of B_0 consisting of all those integers whose k_2-digit (in the mixed radix representation) lies in the set $\{0, 1, \ldots, p_2 - 1\}$. Again as $p_2 \leq q_2$ we have $p_2 - 1 \leq k_2 - 1$ so B' is also well-defined. A similar argument holds when we are considering $A = A_0 + B'$. Here, A is the set of numbers whose k_2-digit is in $\{0, 1, \ldots, p_2 - 1\}$. Thus A consists exactly of the numbers less than or equal to $(p_2 - 1)q_1 \pmod{q_1 q_2}$. This follows as the base of the first digit is q_1. Again it is clear that A has density $(p_2 q_1)/(q_1 q_2) = p_2/q_2$.

By Proposition 16, $r(A, B, n)$ is monotone increasing. □

Finally, we determine for which sets A, B the representation function $r(A, B, n)$ is eventually *strictly* increasing. The corresponding question for a single set has been considered by Chen and Tang [2] who discuss when the functions r, r_1, r_2 are strictly increasing. When considering two sets and the function r, the problem turns out to be easy.

Proposition 19. *Let* $A, B \subset \mathbb{N}_0$*, then the representation function* $r(A, B, n)$ *is eventually strictly monotone increasing if and only if* $\overline{A}$ *and* $\overline{B}$ *are finite.*

Proof. First, let us assume that $r(A,B,n)$ is eventually strictly increasing. We will use the trivial identity that

$$n+1 = r(\mathbb{N}_0,\mathbb{N}_0,n) = r(A,B,n) + r(\overline{A},B,n) + r(A,\overline{B},n) + r(\overline{A},\overline{B},n),$$

which is equivalent to

$$n+1-r(A,B,n) = r(\overline{A},B,n) + r(A,\overline{B},n) + r(\overline{A},\overline{B},n).$$

In the last identity the left hand side is bounded, since we assumed that $r(A,B,n)$ is eventually strictly increasing. Thus so is the right hand side. Hence $r(\overline{A},B,n)$, $r(A,\overline{B},n)$ and $r(\overline{A},\overline{B},n)$ are all bounded. From this it follows that $r(\overline{A},\mathbb{N}_0,n) = r(\overline{A},B,n)+r(\overline{A},\overline{B},n)$ and $r(\mathbb{N}_0,\overline{B},n) = r(A,\overline{B},n)+ r(\overline{A},\overline{B},n)$ are bounded. Thus $\overline{A}$ and $\overline{B}$ must be finite.

Now we assume that $\overline{A}$ and $\overline{B}$ are finite. For any $n > \max(\overline{A}) + \max(\overline{B})$ we know that $a \in \overline{A}$ implies $n-a \notin \overline{B}$ and vice versa, so we can write

$$\begin{aligned} r(A,B,n) &= n+1-|\overline{A}|-|\overline{B}| \\ &< n+2-|\overline{A}|-|\overline{B}| = r(A,B,n+1) \end{aligned}$$

Thus for $n > \max(\overline{A}) + \max(\overline{B})$ the representation function is strictly increasing. □

4. Open Problems

A far-reaching goal would be to completely characterize co-Sidon sets. Which co-Sidon sets are subsets of some perfect co-Sidon sets? Are two random sets likely to be co-Sidon?

Can we completely characterize sets A, B whose representation function is monotone increasing? Are there constructions that do not come from perfect co-Sidon sets?

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