The 3-colored Ramsey Number of Even Cycles

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Abstract

Denote by $R(L, L, L)$ the minimum integer N such that any 3-coloring of the edges of the complete graph on N vertices contains a monochromatic copy of a graph L. Bondy and Erdős conjectured that when L is the cycle C_n on n vertices, $R(C_n, C_n, C_n) = 4n - 3$ for every odd $n > 3$. Łuczak proved that if n is odd, then $R(C_n, C_n, C_n) = 4n + o(n)$, as $n \to \infty$, and Kohayakawa, Simonovits and Skokan confirmed the Bondy-Erdős conjecture for all sufficiently large values of n.

Figaj and Łuczak determined an asymptotic result for the 'complementary' case where the cycles are even: they showed that for even n, we have $R(C_n, C_n, C_n) = 2n + o(n)$, as $n \to \infty$. In this paper, we prove that there exists n_1 such that for every even $n \ge n_1$, $R(C_n, C_n, C_n) = 2n$.

Keywords: Cycles, Ramsey number, Regularity Lemma, Stability

1. Introduction

For graphs $L_1, \ldots, L_k$, the Ramsey number $R(L_1, \ldots, L_k)$ is the minimum integer N such that for any edge-coloring of K_N, the complete graph on N vertices, by k colors, there exists a color i for which the corresponding color class contains L_i as a subgraph. Bondy and Erdős [4] conjectured that if $n > 3$ is odd and L_1, L_2, L_3 are C_n, the cycle on n vertices, then

$$R(C_n, C_n, C_n) = 4n - 3. \tag{1}$$

Łuczak [11] showed that if n is odd, then $R(C_n, C_n, C_n) = 4n + o(n)$, as $n \to \infty$, and Kohayakawa, Simonovits and Skokan [9] proved that there exists an n_0 such that (1) holds for every n odd with $n > n_0$.

The case when n is even differs from the case when n is odd. Figaj and Łuczak [6] proved that for $\alpha_1, \alpha_2, \alpha_3 > 0$,

$$R(C_{2\lfloor \alpha_1 n \rfloor}, C_{2\lfloor \alpha_2 n \rfloor}, C_{2\lfloor \alpha_3 n \rfloor}) = (\alpha_1 + \alpha_2 + \alpha_3 + \max\{\alpha_1, \alpha_2, \alpha_3\} + o(1))n,$$

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as $n \to \infty$. In particular, for n even, we have

$$R(C_n, C_n, C_n) = 2n + o(n), \text{as } n \to \infty.$$

For the path P_n on n vertices this implies that

$$R(P_n, P_n, P_n) = 2n + o(n), \text{as } n \to \infty.$$

Slightly later, independently, Gyárfás, Ruszinkó, Sárközy, and Szemerédi [7] proved a similar but more precise result for paths: there exists an n_0 such that for $n > n_0$,

$$R(P_n, P_n, P_n) = \begin{cases} 2n - 1, & n \text{ odd} \\ 2n - 2, & n \text{ even.} \end{cases}$$

In this paper we prove the following theorem.

Theorem 1. *There exists an integer n_1 such that for every even $n > n_1$,*

$$R(C_n, C_n, C_n) = 2n.$$

Our proof generally follows the proof-line of Gyárfás et al. [7] in which we strengthen some of their lemmas and introduce new ones in order to find monochromatic cycles instead of just paths.

2. Notation

Our notation is standard. For graphs, the first subscripts indicate the number of vertices, e.g., G_n is always a graph of n vertices. C_n is the cycle with n vertices and P_n is the path with n vertices. The *length* of a path is a number of its edges and, if x is its first vertex and x' is its last vertex, then we call it an (x, x')-path. Given a set X of vertices of a graph G, $G[X]$ denotes the subgraph induced by the edges with both ends in X and $G \setminus X$ denotes the subgraph obtained by deleting the vertices of X and the edges incident to the deleted vertices.

Given two disjoint non-empty sets of vertices X and Y, $E(X, Y)$ denotes the set of all the edges with one end in X and the other one in Y. We also set $e(X, Y) = |E(X, Y)|$ and

$$d(X, Y) = \frac{e(X, Y)}{|X||Y|}.$$

We denote the bipartite subgraph of G with bipartition $X \cup Y$ and the edge set $E(X, Y)$ by $G[X, Y]$ and $K(X, Y)$ stands for the complete bipartite graph with bipartition $X \cup Y$.

Whenever we speak about colorings, we mean edge-colorings. Mostly we use three colors, red, blue and green, and the subgraphs induced by the edges of a given color are indicated by superscripts: G^R is the red subgraph of G. However, the corresponding graph theoretical parameters, such as the number of edges or degrees, will be indicated by subscripts: $e_R(X, Y)$ denotes the number of red edges joining X to Y in an edge-colored graph. If an edge xy of G is red, we say that y is a red neighbor of x (and vice-versa). For a vertex x, $N(x)$ denotes the set of all vertices adjacent to x and we set $\deg(x, Y) := |N(x) \cap Y|$ (the degree of x to Y) and $\deg_R(x, Y) := |N_R(x) \cap Y|$ (the red degree of x to Y).

A graph G_n is called γ-dense if it has at least $\gamma\binom{n}{2}$ edges. A bipartite graph with partite sets of size k and ℓ is γ-dense if it contains at least $\gamma k \ell$ edges.

A matching in a graph G is a set of pairwise vertex-disjoint edges. A *connected matching* is a matching M such that all the edges of M are in the same component of G.

3. Extremal colorings and Stability

Below we give a coloring that shows the lower bound (that is, $R(C_n, C_n, C_n) > 2n - 1$) in Theorem 1.

Coloring 1 ($EC_{MAX}(n)$). *Let $n \geq 4$ be even and let $A \cup B \cup C \cup D \cup K$ be a partition of the vertices of K_{2n-1} such that $|A| = |B| = |C| = |D| = n/2 - 1$ and $|K| = 3$ (recall that n is even). Let $K = \{r, g, b\}$. Color the edges inside A, B, C, D arbitrarily, the edges in $E(A, B) \cup E(C, D)$ by red, the edges in $E(A, D) \cup E(B, C)$ by green, and the edges in $E(A, C) \cup E(B, D)$ by blue. Now color the edges from r to $A \cup B \cup C \cup D$ by red and the edges from g to $A \cup B \cup C \cup D \cup \{r\}$ by green. Finally, color all the edges incident to b by blue.*

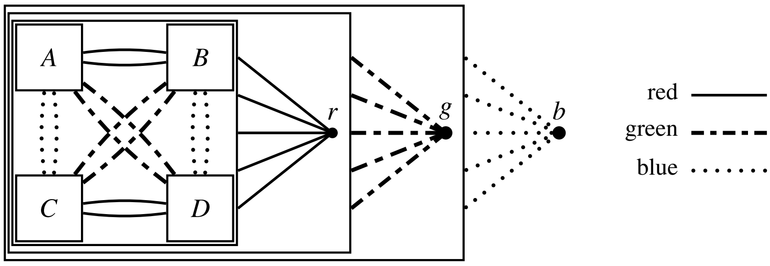

Figure 1: Coloring of K_{2n-1} with no C_n

Lemma 2. *For all $n \geq 4$ even, $R(C_n, C_n, C_n) > 2n - 1$.*

Proof. We must show that any $n \geq 4$ even, coloring $EC_{MAX}(n)$ does not contain any monochromatic C_n. Let G^B, G^R, G^G be the color classes of $EC_{MAX}(n)$. It is clear that $G_1^R := G^R \setminus \{r, g, b\}$, $G_1^B := G^B \setminus \{r, g, b\}$ $G_1^G := G^G \setminus \{r, g, b\}$ do not contain any monochromatic C_n because each of their components has order $n - 2 < n$. Since r is the only vertex that is adjacent (in G^R) to the both components of G_1^R, there is no red C_n in G^R. Similarly, there are no monochromatic C_n in G^B or G^G. HERE □

To prove the upper bound in Theorem 1, i.e., $R(C_n, C_n, C_n) \leq 2n$, we will need to look at another three types of colorings. It will be also convenient to consider multi-3-colorings instead of 3-colorings. In a *multi-3-coloring* of a graph G, some of its edges can be assigned more than one color. We say that an edge is *C-exclusive* (or *exclusive in color C*) for $C \in$ {(R)ed, (G)reen, (B)lue} if it is assigned color C only. We denote by G^{C^*} the subgraph induced by the C-exclusive edges. Now we define the 3 types of colorings:

Coloring 2 ($EC_1(\alpha, \delta)$ type).
A (multi-3-)coloring of a graph G is of type $EC_1(\alpha, \delta)$, where $0 \leq \alpha, \delta < 1$, if there exists a partition $A \cup B \cup C \cup D$ of $V(G)$ such that

(a) $|A|, |B|, |C|, |D| \geq (1 - \alpha)|V(G)|/4$;
(b) *The bipartite graphs $G^{R^*}[A, B], G^{R^*}[C, D], G^{G^*}[A, D], G^{G^*}[B, C], G^{B^*}[A, C]$ and $G^{B^*}[B, D]$ are $(1 - \delta)$-dense.*

Coloring 3 (***EC*$_2(\alpha, \delta)$** type)**.**
A (multi-3-)coloring of a graph G is of type $EC_2(\alpha, \delta)$*, where* $0 \le \alpha, \delta < 1$*, if there exists a partition* $A \cup B \cup C \cup D$ *of* $V(G)$ *such that*

(a) $|A|, |B|, |C|, |D| \ge (1 - \alpha)|V(G)|/4$;
(b) *The bipartite graphs* $G^{R^*}[A, B]$, $G^{G^*}[A \cup B, C]$ *and* $G^{B^*}[A \cup B, D]$ *are* $(1 - \delta)$*-dense.*

Coloring 4 (***EC*$_3(\mu, c_1, c_2, \delta)$** type)**.**
A (multi-3-)coloring of a graph G is of type $EC_3(\mu, c_1, c_2, \delta)$*, where* $0 \le \mu,\ c_1,\ c_2,\ \delta < 1$*, if there exists a partition* $A \cup B \cup C \cup D$ *of* $V(G)$ *such that*

(a) $|A|, |B|, |C| \ge (1 - c_1\mu)|V(G)|/4, \quad |D| \ge \mu|V(G)|/4$;
(b) $|A| \ge \max\{|B|, |C|, |D|\} + \mu|V(G)|/4$, $|A \cup D| \le (1 + c_2\mu)|V(G)|/2$;
(c) *The bipartite graphs* $G^{R^*}[A, B]$, $G^{R^*}[C, D]$, $G^{G^*}[A, D]$, $G^{G^*}[B, C]$, $G^{B^*}[A, C]$ *and* $G^{B^*}[B, D]$ *are* $(1 - \delta)$*-dense.*

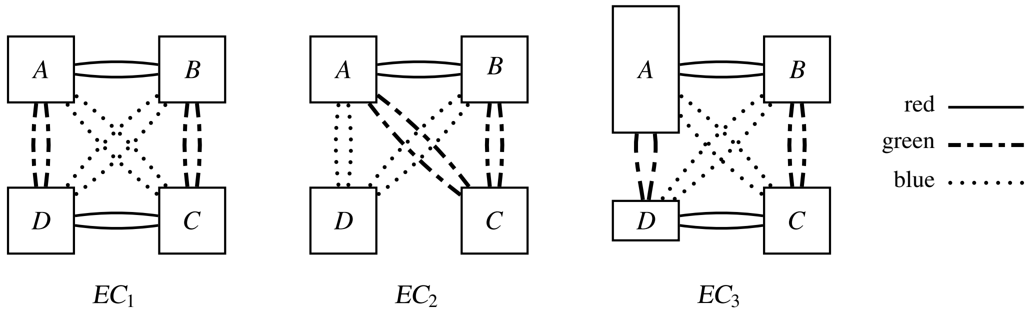

Figure 2: Three different types of colorings

We distinguish the case $\delta = 0$ by giving the above colorings special names.

Definition 3. *We say that a coloring is* $EC_1(\alpha)$*-complete if it is of the type* $EC_1(\alpha, 0)$*, that is, if the monochromatic bipartite graphs involved in the definition of* EC_1 *are complete. We define* $EC_2(\alpha)$*-complete and* $EC_3(\mu, c_1, c_2)$*-complete colorings in a similar way.*

The main tool to prove the upper bound in Theorem 1 is the following variant of the stability theorem proved in [7], [8].

Theorem 4. *Given* $\alpha_1 > 0$ *and* $\mu_1 > 0$*, there exist positive reals* η_4, ϵ_4 *and* μ_4*,* $\mu_4 < \mu_1$*, such that for all* $\epsilon < \epsilon_4$ *there exists a positive integer* n_4 *such that the following holds:*

If $n \ge n_4$ *and a* $(1 - \epsilon)$*-dense graph* G_n *is* 3*-multi-colored, then one of the following cases occurs:*

Case 1: G_n *contains a monochromatic connected matching of size at least* $(1/4 + \eta_4)n$*;*

Case 2: *the coloring is of type* $EC_1(\alpha_1/2, \alpha_1/2)$*;*

Case 3: *the coloring is of type* $EC_2(\alpha_1/2, \alpha_1/2)$*;*

Case 4: *the coloring is of type* $EC_3(\mu_4,\ 0.7,\ 0.2,\ \varepsilon^{1/3})$.

The proof of Theorem 4 is essentially the same as in [7] and it can be found in [1]. In order to deal with Cases 2–4, we will need the following lemmas whose proofs appear in Sections 7 and 8.

Lemma 5. *There exists $\alpha_5 > 0$ such that, for all $\alpha \le \alpha_5$ and all $\delta \le \alpha$, there exists a positive integer n_5 with the following property: For every even $n \ge n_5$, every 3-coloring of K_{2n} of type $EC_1(\alpha, \delta)$ contains a monochromatic C_n.*

Lemma 6. *There exists $\alpha_6 > 0$ such that, for all $\alpha \le \alpha_6$ and all $\delta \le \alpha$, there exists a positive integer n_6 with the following property: For every even $n \ge n_6$, every 3-coloring of K_{2n} of type $EC_2(\alpha, \delta)$ has a monochromatic C_n.*

Lemma 7. *There is an integer $\mu_7 > 0$ such that, for all $\mu \le \mu_7$, $c_1 < 1$ and $c_2 < 0.5$, there exist $n_7 = n_7(\mu, c_1, c_2)$ and $\delta_7 = \delta_7(\mu, c_1, c_2)$ such that the following holds: For every even $n \ge n_7$ and $0 < \delta \le \delta_7$, every 3-coloring of K_{2n} of type $EC_3(\mu, c_1, c_2, \delta)$ contains a monochromatic C_n.*

The remainder of this paper is organized as follows: we present Szemerédi's regularity lemma in the next section. The proof of Theorem 1 is given in Section 5, and Sections 7 and 8 contain the proofs of the above three lemmas.

4. Regularity Lemma for graphs

Szemerédi's regularity lemma [12] asserts that each graph of positive edge-density can be approximated by the union of a bounded number of random-like bipartite graphs. Before it can be stated formally, the concept of *ε-regular pairs* needs to be defined.

Definition 8. *Let $G = (V, E)$ be a graph and let $0 < \varepsilon \le 1$. We say that a pair (A, B) of two disjoint subsets of V is ε-regular (with respect to G) if*

$$|d(A', B') - d(A, B)| < \varepsilon$$

holds for any two subsets $A' \subset A$, $B' \subset B$ with $|A'| > \varepsilon|A|$, $|B'| > \varepsilon|B|$.

This definition states that a regular pair has uniformly distributed edges. In the next section, we will make a use of the following properties of regular pairs.

Fact 9. *Let G be a bipartite graph with bipartition $V(G) = V_1 \cup V_2$ such that the pair (V_1, V_2) is ε-regular with density $d := d(V_1, V_2)$. Then all but at most $\varepsilon|V_1|$ vertices $v \in V_1$ satisfy $\deg(v) \ge (d - \varepsilon)|V_2|$.*

The next lemma about regular pairs is a slightly stronger version of Claim 3 in [11]. The version in [11] is the case where $\beta = 1$. Both statements have the same proof that we omit here.

Lemma 10. *For every $0 < \beta < 1$ there exists an n_{10} such that for every $n > n_{10}$ the following holds: Let G be a bipartite graph with bipartition $V(G) = V_1 \cup V_2$ such that $|V_1|, |V_2| \ge n$. Furthermore let the pair (V_1, V_2) be ε-regular with density at least $\beta/4$ for some ε satisfying $0 < \varepsilon < \beta/100$. Then for every ℓ, $1 \le \ell \le n - 5\varepsilon n/\beta$, and for every pair of vertices $v' \in V_1$, $v'' \in V_2$ satisfying $\deg(v')$, $\deg(v'') \ge \beta n/5$, G contains a path of length $2\ell + 1$ connecting v' and v''.*

The regularity lemma of Szemerédi [12] enables us to partition the vertex set $V(G)$ of a graph G into $t+1$ sets $V_0 \cup V_1 \cup \ldots \cup V_t$ in such a way that almost all the pairs (V_i, V_j) satisfy Definition 8. Its precise statement, extended to more than one graph, is as follows.

Theorem 11. *For every $\varepsilon > 0$ and $s, m \in \mathbb{N}$ there exist integers $N_{11} = N_{11}(\varepsilon, s, m)$ and $M_{11} = M_{11}(\varepsilon, s, m)$ such that: for all graphs $G_1, \ldots, G_s$ with the same vertex set V, $|V| \geq N_{11}$, there is a partition of V into $t + 1$ sets*

$$V = V_0 \cup V_1 \cup \ldots \cup V_t$$

such that

(a) *$m \leq t \leq M_{11}$,*
(b) *$|V_0| \leq \varepsilon n$, $|V_1| = \ldots = |V_t|$, and*
(c) *all but at most $\varepsilon\binom{t}{2}$ pairs (V_i, V_j), $1 \leq i < j \leq t$, are ε-regular with respect to each G_k, $1 \leq k \leq s$.*

Remark 12. *The original regularity lemma refers to the case $s = 1$. The proof is (basically) the same for an arbitrary but fixed number s of graphs. This version is used, for example, in [5], and formulated in the survey [10].*

5. Proof of Theorem 1

We give first a brief overview of the proof. We start the proof by defining a number of parameters in order to be able to apply a sequence of lemmas later. At this point (see (3)), we will also choose n_1 (that is, the absolute constant from Theorem 1).

Next, we consider a 3-coloring (G^R, G^G, G^B) of the complete graph K_{2n}, where n is even and $n > n_1$. We apply the regularity lemma (Theorem 11) with carefully chosen ε (see (2)) to G^R, G^G, G^B and obtain a partition $V_0 \cup V_1 \cup \cdots \cup V_t$ of $V(K_{2n})$ satisfying conditions (a)-(c) in Theorem 11. Using this partition we define the so-called reduced graph H and also an appropriate multi-3-coloring of its edges: The vertex set of H is $\{1, \ldots, t\}$, we have an edge between i and j if and only if (V_i, V_j) is an ϵ-regular pair with respect to G^R, G^G and G^B, and an edge ij is is colored by red (blue, green, respectively) if $G^R[V_i, V_j]$ ($G^B[V_i, V_j]$, $G^G[V_i, V_j]$, respectively) has the edge density at least $\varepsilon^{1/3}/4$.

Then, we apply Theorem 4 to H, which will lead us to one of four cases: either H has a monochromatic connected matching of a certain size or its multi-3-coloring is of type EC_1 or EC_2 or EC_3. In the first case, we use the monochromatic matching in H to find a copy of C_n of the same color in K_{2n} by applying Lemma 10. In other three cases, we prove that the original coloring of K_{2n} must be of the same type as the coloring of H. Then we apply Lemma 5, Lemma 6 or Lemma 7 to find a monochromatic C_n in K_{2n}.

Proof. We have already proved the lower bound in Lemma 2. Let us prove the upper bound. Set $\alpha_1 := \min\{\alpha_5, \alpha_6, 1/20\}$ so that, in particular, we can use α_1 as an input for Lemmas 5 and 6 and obtain $n_5 = n_5(\alpha_1, \alpha_1)$ and $n_6 = n_6(\alpha_1, \alpha_1)$. Passing α_1 and $\mu_1 := \mu_7$ (the absolute constant from Lemma 7) as parameters to Theorem 4, we obtain ϵ_4, η_4 and $\mu_4 < \mu_7$. Let

$$\eta := \eta_4.$$

Now, inputting $\mu := 0.99\mu_4$, $c_1 := 0.8$, $c_2 := 0.3$ into Lemma 7, we obtain n_7 and δ_7. We define

$$\varepsilon := \frac{1}{2} \min\left\{\varepsilon_4, \frac{\delta_7^3}{8}, \frac{1}{10^6}, \frac{\alpha_1^3}{1000}, \frac{\mu_4}{100}\right\}. \tag{2}$$

This choice particularly means that $\varepsilon < \varepsilon_4$, $2\varepsilon^{1/3} < \delta_7$ and $\varepsilon < 0.001\varepsilon^{1/3}$, all of which we will use later. For this ε, Lemma 4 yields n_4 and Lemma 10 applied with $\beta = \varepsilon^{1/3}$ gives n_{10}. Now we may chose $m = \max\{n_4, 1/\varepsilon\}$ and, from Theorem 11, we obtain $N_{11} = N_{11}(\varepsilon, 3, m)$ and $M_{11} = M_{11}(\varepsilon, 3, m)$. Finally, define

$$n_1 = \max\left\{n_5, n_6, n_7, N_{11}, 4M_{11}n_{10}, \frac{M_{11}^2}{\varepsilon}\right\}. \tag{3}$$

Consider any 3-coloring (G^R, G^G, G^B) of K_{2n} with n even and $n > n_1$. We apply the regularity lemma with parameters ϵ, m and $s := 3$ to G^R, G^G, G^B. Let $V := V(K_{2n}) = V_0 \cup V_1 \cup \ldots \cup V_t$ be the partition guaranteed by this lemma, thus satisfying

(a) $m \leq t \leq M_{11}$,
(b) $|V_0| \leq \varepsilon(2n)$, $|V_1| = \ldots = |V_t|$, and
(c) all but at most $\varepsilon\binom{t}{2}$ pairs (V_i, V_j), $1 \leq i < j \leq t$, are ε-regular with respect to each G^R, G^G, G^B.

Now we define the *reduced graph* H in the following way: the vertex set of H is $\{1, \ldots, t\}$ and we have an edge between i and j if and only if (V_i, V_j) is an ϵ-regular pair with respect to G^R, G^G and G^B. Notice that by (c),

$$e(H) \geq (1 - \epsilon)\binom{t}{2},$$

that is, H is $(1 - \varepsilon)$-dense.

We define a 3-multi-coloring (H^R, H^G, H^B) of H in the following way: for $c \in \{R, B, G\}$, we put the edge ij into H^c if $e_c(V_i, V_j) \geq \varepsilon^{1/3}|V_i||V_j|/3$. Since $t \geq m \geq n_4$, we can apply Theorem 4 to H and distinguish four cases.

Case 1: *There is a monochromatic connected matching M of size $t_1 \geq (1/4 + \eta)t$ in H.* Without loss of generality assume that M is red and let a_ib_i, $0 \leq i < t_1$, be all the edges of M.

Now, we will use standard regularity arguments to built a (red) cycle of length n in G^R. First, let F be any minimal connected red subgraph of H containing M. Clearly, F is a tree. Consider a closed minimal walk $W = i_1i_2 \ldots i_\ell i_1$ that contains all the edges of F. Since F is a tree, W must be of even length and $\ell \leq 2t$.

Using Fact 9 repeatedly, we find an even red cycle $\tilde{C} = v_{i_1}v_{i_2} \ldots v_{i_\ell}$ such that $v_{i_j} \in V_{i_j}$ and v_{i_j} has at least $\varepsilon^{1/3}|V_{i_{j-1}}|/4$ red neighbors in $V_{i_{j-1}}$ and at least $\varepsilon^{1/3}|V_{i_{j+1}}|/4$ red neighbors in $V_{i_{j+1}}$ for all $j = 1, \ldots, \ell$. We emphasize that while we may have $V_{i_k} = V_{i_\ell}$, for some $k \neq \ell$, the vertices v_{i_j} of C are chosen to be pairwise distinct. (We set $V_{i_0} := V_{i_\ell}$ and $V_{i_{\ell+1}} := V_{i_1}$.)

Then, for each edge a_kb_k of M, we choose a natural number ℓ_k satisfying

$$1 \leq \ell_k \leq \left(1 - \frac{5\varepsilon}{\varepsilon^{1/3}}\right)\min\{|V_{a_k}| - 2t, |V_{b_k}| - 2t\}$$

in such a way that

$$\sum_{k=0}^{t_1-1} 2\ell_k = n - \ell.$$

This is possible because $n - \ell$ is even, $n > n - \ell \geq n - 2t \geq 2t \geq 2t_1$, and $\sum_{k=0}^{t_1-1} 2\ell_k$ can attain any even value between $2t_1$ and

$$\begin{aligned}\sum_{i=0}^{t_1-1} 2\left(1 - \frac{5\varepsilon}{\varepsilon^{1/3}}\right)\min\{|V_{a_k}| - 2t, |V_{b_k}| - 2t\} &\geq 2t_1\left(1 - \frac{5\varepsilon}{\varepsilon^{1/3}}\right)\left(\frac{(1-\varepsilon)2n}{t} - 2t\right) \\ &\geq \left(\frac{1}{2} + 2\eta\right)t(1 - 5\varepsilon^{2/3})\frac{(1-2\varepsilon)2n}{t} \\ &\geq (1 + 3\eta)n.\end{aligned}$$

Finally, we set $V'_{a_k} = (V_{a_k} \setminus \tilde{C}) \cup \{v_{a_k}\}$, $V'_{b_k} = (V_{b_k} \setminus \tilde{C}) \cup \{v_{b_k}\}$ and notice that

$$|V'_{a_k}| \geq |V_{a_k}| - |\tilde{C}| \geq |V_{a_k}| - 2t \geq \frac{(1-\varepsilon)2n}{t} - 2t \geq \frac{(1-2\varepsilon)2n}{t} \geq \frac{|V_{a_k}|}{2} \geq \frac{(1-\varepsilon)2n}{2M_{11}} > n_{10}$$

and, similarly,

$$|V'_{b_k}| \geq \frac{|V_{b_k}|}{2} > n_{10}.$$

Hence, $G^R[V'_{a_k}, V'_{b_k}]$ is (2ε)-regular with density at least $\varepsilon^{1/3}/4 - \varepsilon > \varepsilon^{1/3}/5$ and we can apply Lemma 10 to $G^R[V'_{a_k}, V'_{b_k}]$. Since

$$1 \leq \ell_k \leq \left(1 - \frac{5\varepsilon}{\varepsilon^{1/3}}\right)\min\{|V_{a_k}| - 2t, |V_{b_k}| - 2t\} \leq \left(1 - \frac{5\varepsilon}{\varepsilon^{1/3}}\right)\min\{|V'_{a_k}|, |V'_{b_k}|\},$$

there exists a path P_{a_k,b_k} of length $2\ell_k + 1$ that starts at v_{a_k}, ends at v_{b_k}, and consists only of edges in $G^R[V'_{a_k}, V'_{b_k}]$. In $\tilde{C}$, we replace each edge $v_{a_k}v_{b_k}$ by the path P_{a_k,b_k}. This yields a red cycle of length $\ell - t_1 + \sum_{k=0}^{t_1-1}(2\ell_k + 1) = n$.

Case 2: (H^R, H^G, H^B) *is a coloring of type* $EC_1(\alpha_1/2, \alpha_1/2)$. We will show that this implies that (G^R, G^G, G^B) is of type $EC_1(\alpha_1, \alpha_1)$. Let $A \cup B \cup C \cup D$ be a partition of $V(H)$ satisfying conditions (a) and (b) of $EC_1(\alpha_1/2, \alpha_1/2)$, and consider partition $(f(A) \cup V_0) \cup f(B) \cup f(C) \cup f(D)$ of V, where $f(X) := \bigcup_{i \in X} V_i$.

First note that

$$|f(A) \cup V_0| \geq |A|\frac{(1-\epsilon)(2n)}{t} \geq \left(1 - \frac{\alpha_1}{2}\right)\frac{t}{4}\frac{(1-\epsilon)(2n)}{t} \geq (1 - \alpha_1)\frac{2n}{4}.$$

Similarly, we obtain that $|f(B)|, |f(C)|, |f(D)| \geq (1 - \alpha_1)2n/4$ as well. Hence, condition (a) of $EC_1(\alpha_1, \alpha_1)$ holds.

Next we show that condition (b) in $EC_1(\alpha_1, \alpha_1)$ is also true for this partition $(f(A) \cup V_0) \cup f(B) \cup f(C) \cup f(D)$ of V and the original 3-coloring (G^R, G^G, G^B) of K_{2n}. Note that since there are no multicolored edges in K_{2n}, we have $G^{R^*} = G^R$, $G^{B^*} = G^B$ and $G^{G^*} = G^G$. We first estimate the number of edges between $f(A) \cup V_0$ and $f(B)$ that are not in $G^{R^*} = G^R$.

For every $i \in A, j \in B$ such that $ij \notin H^{R^*}$, the number of edges in $K(V_i, V_j) \cap (G^G \cup G^B)$ is bounded by $|V_i||V_j|$. On the other hand, for each edge $ij \in H^{R^*}$ we have that $ij \notin H^G \cup H^B$, thus, by the definition of H^G and H^B, the number of edges in $K(V_i, V_j) \cap (G^G \cup G^B)$ is bounded by $2\left(\varepsilon^{1/3}|V_i||V_j|/4\right)$. We have no information about the edges from V_0 to $f(B)$, so we can only estimate the number of these edges by $|V_0||f(B)|$. Hence,

$$\begin{aligned}|K(f(A)\cup V_0, f(B))\cap(G^G\cup G^B)| \quad &\leq \sum_{\substack{i\in A, j\in B\\ ij\notin H^{R^*}}}|V_i||V_j| + \sum_{\substack{i\in A, j\in B\\ ij\in H^{R^*}}} 2\frac{\varepsilon^{1/3}}{4}|V_i||V_j| \\ &+|V_0||f(B)|.\end{aligned}$$

We estimate all three terms on the right-hand side. There are at most $\alpha_1|A||B|/2$ pairs $i\in A$, $j\in B$ such that $ij\notin H^{R^*}$. We also know that $|V_1|=\cdots=|V_t|=(2n-|V_0|)/t$. Hence,

$$\sum_{\substack{i\in A, j\in B\\ ij\notin H^{R^*}}}|V_i||V_j| \leq \frac{\alpha_1}{2}|A||B|\left(\frac{2n-|V_0|}{t}\right)^2 = \frac{\alpha_1}{2}\sum_{i\in A, j\in B}|V_i||V_j| = \frac{\alpha_1}{2}|f(A)||f(B)|.$$

Since $\varepsilon^{1/3}<\alpha_1/100$, $|V_0|\leq\varepsilon(2n)$ and $|f(A)\cup V_0|\geq(1-\alpha_1)2n/4$, we have

$$\sum_{\substack{i\in A, j\in B\\ ij\in H^{R^*}}} 2\frac{\varepsilon^{1/3}}{4}|V_i||V_j| \leq \frac{\varepsilon^{1/3}}{2}\sum_{i\in A, j\in B}|V_i||V_j| \leq \frac{\alpha_1}{4}|f(A)||f(B)|$$

and

$$|V_0||f(B)| \leq \varepsilon(2n)|f(B)| \leq 5\varepsilon(1-\alpha_1)\frac{2n}{4}|f(B)| \leq \frac{\alpha_1}{4}|f(A\cup V_0)||f(B)|.$$

Consequently,

$$|K(f(A)\cup V_0, f(B))\cap(G^G\cup G^B)| \leq \alpha_1|f(A)\cup V_0||f(B)|$$

and $K(f(A)\cup V_0, f(B))\cap G^{R^*}$ is $(1-\alpha_1)$-dense.

In the same way, one proves that the bipartite graphs $G^{R^*}[f(C), f(D)]$, $G^{G^*}[f(A)\cup V_0, f(D)]$, $G^{G^*}[f(B), f(C)]$, $G^{B^*}[f(A)\cup V_0, f(C)]$ and $G^{B^*}[f(B), f(D)]$ are all $(1-\alpha_1)$-dense. We omit the technical details here.

So, the given 3-coloring of K_{2n} is of type $EC_1(\alpha_1,\alpha_1)$, $n>n_5$ and $\alpha_1<\alpha_5$. We use Lemma 5 to conclude that there is a monochromatic C_n in K_{2n}.

Case 3: (H^R, H^G, H^B) *is of type* $EC_2(\alpha_1/2, \alpha_1/2)$. Similarly to the previous case, one can show that (G^R, G^G, G^B) is of type $EC_2(\alpha_1,\alpha_1)$. Since $n>n_6$ and $\alpha_1<\alpha_6$, we use Lemma 6 and find the monochromatic C_n in K_{2n}. We omit technical details.

Case 4: (H^R, H^G, H^B) *is of type* $EC_3(\mu_4,\ 0.7,\ 0.2,\ \varepsilon^{1/3})$. We claim that in this case, (G^R, G^G, G^B) is of type $EC_3(0.99\mu_4, 0.8, 0.3, 2\varepsilon^{1/3})$. Indeed, let $A\cup B\cup C\cup D$ be a partition of $V(H)$ such that conditions (a)-(c) of $EC_3(\mu_4,\ 0.7,\ 0.2,\ \varepsilon^{1/3})$ hold and consider partition $(f(A)\cup V_0)\cup f(B)\cup f(C)\cup f(D)$ of V, where $f(X):=\bigcup_{i\in X}V_i$. Clearly,

$$|f(D)| \geq |D|\frac{(1-\epsilon)(2n)}{t} \geq \mu_4\frac{t}{4}\frac{(1-\epsilon)(2n)}{t} \geq (1-\epsilon)\mu_4\frac{2n}{4} \geq 0.99\mu_4\frac{2n}{4}.$$

Furthermore,

$$|f(A)\cup V_0|, |f(B)|, |f(C)| \geq (1-0.7\mu_4)\,\frac{t}{4}\frac{(1-\epsilon)(2n)}{t} \geq (1-0.8(0.99\mu_4))\frac{2n}{4},$$

hence, condition (a) of $EC_3(0.99\mu_4, 0.8, 0.3, 2\varepsilon^{1/3})$ is true. Also notice that if $|A| - |B| \geq \mu_4 t/4$, then

$$|f(A)| - |f(B)| \geq \mu_4 \frac{t}{4} \frac{(1-\varepsilon)(2n)}{t} = (1-\varepsilon)\mu_4 \frac{2n}{4} \geq 0.99\mu_4 \frac{2n}{4},$$

and the same holds if we replace B with C or D. It follows that

$$|f(A) \cup V_0| \geq |f(A)| \geq \max\{|f(B)|, |f(C)|, |f(D)|\} + 0.99\mu_4 \frac{2n}{4}.$$

Finally, since $\varepsilon \leq \mu_4/100$, we get

$$|f(A) \cup V_0 \cup D| \leq (1 + 0.2\mu_4) \frac{t}{2} \frac{2n}{t} + \varepsilon(2n) \leq (1 + 0.3(0.99\mu_4)) \frac{2n}{2}.$$

Thus, condition (b) of $EC_3(0.99\mu_4, 0.8, 0.3, 2\varepsilon^{1/3})$ holds as well. Condition (c) can be verified in a similar way as in Case 2.

To finish this case, we use Lemma 7 with $\mu = 0.99\mu_4$ and $\delta = 2\varepsilon^{1/3}$ to find the monochromatic C_n in K_{2n}. The assumptions of this lemma are satisfied because $0.99\mu_4 < \mu_4 < \mu_7$, $2\varepsilon^{1/3} < \delta_7(0.99\mu_4, 0.8, 0.3)$ and $n \geq n_7$. □

6. Paths and cycles in (bipartite) graphs

In our proof of Lemmas 5 and 7, we will need the following well-known facts.

Theorem 13 ([3]). *Suppose that H_n is a graph with minimum degree bigger than $n/2$. Then H_n contains the cycle C_k for each $k = 3, \ldots, n$.*

Lemma 14 ([2], page 107). *Let H_n be a graph containing no P_{k+1}, $k \geq 1$. Then $e(H_n) \leq (k-1)n/2$. Furthermore, if $e(H_n) = (k-1)n/2$, then H_n is the disjoint union of cliques K_k.*

The next 3 lemmas are from [1] and their proofs are based on greedy (embedding) algorithm.

Lemma 15 (Lemma 5.7 in [1]). *Let $0 \leq \beta < 1/4$ and let H be a bipartite graph with bipartition $X \cup Y$, $|X|, |Y| \geq 4$, such that for every $x \in X$, $\deg(x, Y) \geq (1-\beta)|Y|$ and for every $y \in Y$, $\deg(y, X) \geq (1-\beta)|X|$. Then*

(a) *for any two vertices $x, x' \in X$ there exists an (x, x')-path of length $2k-2$ for every $2 \leq k \leq \min\{|X|, (1-2\beta)|Y|\}$; the analogous statement, obtained by exchanging the two vertex classes, also holds;*
(b) *for any two vertices $x \in X$, $y \in Y$ there exists an (x, y)-path of length $2k-1$ for every odd $2 \leq k \leq (1-2\beta)\min\{|X|, |Y|\}$.*

Proof. In order to prove (a), we first pick k distinct vertices $x_1, \ldots, x_k \in X$ (recall that $k \leq |X|$) such that $x_1 = x$ and $x_k = x'$. Then we build inductively a path $P_k = x_1 y_1 x_2 y_2 \ldots y_{k-1} x_k$, with $y_i \in Y$ for all i, $1 \leq i \leq k-1$. Assuming that for a given ℓ, $1 \leq \ell \leq k-1$, we have built $P_\ell = x_1 y_1 \ldots y_{\ell-1} x_\ell$, let y_ℓ be any vertex in the common neighborhood of $x_{\ell-1}$ and x_ℓ which is not in $V(P_\ell)$. Then set $P_{\ell+1} := P_\ell y_\ell x_{\ell+1}$. Such a vertex must exist because

$$\left|(N(x_{\ell-1}) \cap N(x_\ell)) \setminus V(P_\ell)\right| \geq (|B| - 2\beta|B|) - (\ell - 1) \geq 2,$$

where the last inequality follows from the fact that

$$\ell \leq k - 1 \leq (1 - 2\beta)|B| - 1.$$

The proof of (b) is similar: first take any neighbor x' of y such that $x' \neq x$, and then apply the previous construction to find a path of length $2k - 2$ from x to x', while making sure that this path also avoids y. □

Lemma 16 (Lemma 5.8 in [1]). *Let $0 \leq \beta < 1/3$ and let H be a bipartite graph with bipartition $\tilde{X} \cup \tilde{Y}$ where, $|\tilde{X}| = |\tilde{Y}|$. Suppose that $X' \cup X$ is a partition of $\tilde{X}$ and $Y' \cup Y$ is a partition of $\tilde{Y}$ such that*

(a) $|X'| \leq \beta|\tilde{X}|$, $|Y'| \leq \beta|\tilde{Y}|$;
(b) *$H[X, Y]$ is the complete bipartite graph;*
(c) *there is an ordering $x_1, \ldots, x_k$ of X' such that* $\deg(x_i, Y) \geq 2i$ *for all* $1 \leq i \leq k$;
(d) *there is an ordering $y_1, \ldots, y_\ell$ of Y' such that* $\deg(y_i, X) \geq 2i$ *for all* $1 \leq i \leq \ell$.

Then H is Hamiltonian.

Proof. Suppose that all the conditions in the lemma hold and let $x_1, \ldots, x_k$, $y_1, \ldots, y_\ell$ be the vertices given by (c) and (d).

We may assume $k = \ell$, i.e. $|X'| = |Y'| \leq \beta|\tilde{X}| = \beta|\tilde{Y}|$, by means of sending vertices from X to X' or from Y to Y' if necessary. (For that we use that $\beta < 1/3$). By condition (c), for each x_i, $1 \leq i \leq k$, we can choose two of its neighbors, say $a_{i,1}, a_{i,2} \in Y$, such that $a_{1,1}, \ldots, a_{k,1}$, $a_{1,2}, \ldots, a_{k,2}$ are pairwise distinct. Similarly, by (d), for $1 \leq i \leq k$, we can pick neighbors $b_{i,1}, b_{i,2} \in X$ of y_i, such that $b_{1,1}, \ldots, b_{k,1}, b_{1,2}, \ldots, b_{k,2}$ are pairwise distinct.

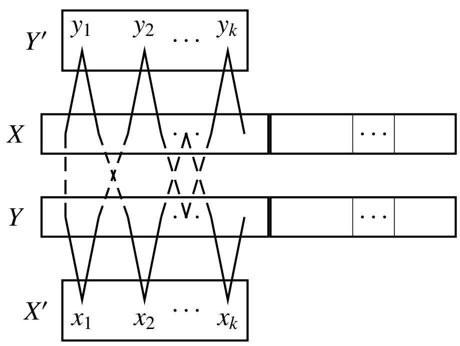

Figure 3: Construction of the path P

Since the graph $H[X, Y]$ is complete, it contains the edges $a_{1,1}b_{1,1}$ and $a_{i,2}b_{i+1,1}$, $b_{i,2}a_{i+1,1}$ for every i, $1 \leq i \leq k - 1$. Hence we get an $(a_{k,2}, b_{k,2})$-path P of order $6k$, with vertex set $V(P) = \{x_i, y_i, a_{i,j}, b_{i,j} \mid 1 \leq i \leq k, 1 \leq j \leq 2\}$. Since $H \setminus V(P)$ is a complete bipartite graph with equally sized parts, it is trivial to extend P to a hamiltonian cycle. □

Lemma 17 (Lemma 5.9 in [1]). *Let $0 \leq \beta < 1/4$, $n, t \in \mathbb{N}$, let H be a graph, and let $\tilde{X}$, $\tilde{Y}$ be two disjoint subsets of $V(H)$ satisfying $|\tilde{X}| = n/2 + t$ and $|\tilde{Y}| = n/2 - t$. Suppose that $X' \cup X$ is a partition of $\tilde{X}$ and $Y' \cup Y$ is a partition of $\tilde{Y}$ such that*

(a) $|X'| \leq \beta(n/2 - t)$, $|Y'| \leq \beta(n/2 - t)$;
(b) $H[X, Y]$ *is the complete bipartite graph;*
(c) *there is an ordering* $x_1, \ldots, x_k$ *of* X' *such that* $\deg(x_i, Y) \geq 2i$ *for all* $1 \leq i \leq k$;
(d) $H[X]$ *contains a path* $P := P_{2t+1}$,
(e) *there is an ordering* $y_1, \ldots, y_\ell$ *of* Y' *such that* $\deg(y_i, X \setminus V(P)) \geq 2i$ *for all* $1 \leq i \leq \ell$.

Then $H[\tilde{X} \cup \tilde{Y}]$ *contains* C_n.

Proof. Let H be a graph and P be a path satisfying all the conditions of the lemma and denote by $u, v \in X$ the endpoints of P. Consider the graph H' obtained from the graph $H[(\tilde{X} \setminus V(P)) \cup \{u\}, \tilde{Y}]$ by removing all the edges from u to Y'. Notice that $|(\tilde{X} \setminus P) \cup \{u\}| = |\tilde{Y}| = n/2 - t$. By the previous lemma, there exists a hamiltonian cycle C in H'. Since u has no neighbors in Y' in the graph H', the neighbors of u in C are in Y. Take one of those neighbors, say u', and replace the edge uu' of C by the path $uPvu'$. The result is a hamiltonian cycle in the original graph H. □

Remark 18. *If, in the last two lemmas, we assume that* $\deg(x, Y) \geq 2\beta|\tilde{X}| \geq 2|X'|$ *for every* $x \in X'$*, then any ordering of the vertices of* X' *satisfies condition (c). Similarly, if* $\deg(y, X) \geq 2|Y'|$ *for every* $y \in Y'$*, then any ordering of the vertices of* Y' *satisfies condition (d).*

7. Proof of Lemmas 5 and 7

Since colorings $EC_1(\alpha, \delta)$ and $EC_3(\mu, c_1, c_2, \delta)$ are quite similar, it turns out we can prove both lemmas simultaneously. The proof is rather long due to the fact that we need to distinguish several sub-cases.

Proof. We set
$$\alpha_5 = \mu_7 := 10^{-12}$$
and, in Lemma 7, for $\mu \leq \mu_7$, define
$$\delta_7 = \min\left\{\left(\frac{\mu}{25}\right)^2, \frac{(1-c_1)^3\mu^3}{10^6}, \left(\frac{(0.5-c_2)\mu}{100}\right)^2\right\}.$$
For $\delta \leq \alpha \leq \alpha_5$, we put
$$n_5 = \lceil \delta^{-12} \rceil$$
and, for $\delta \leq \delta_7$, we put
$$n_7 = \lceil \delta^{-12} \rceil.$$
Consider any 3-coloring of $G = K_{2n}$ of either type $EC_1(\alpha, \delta)$, in which case we assume $n \geq n_5$, or $EC_3(\mu, c_1, c_2, \delta)$, in which case we assume $n \geq n_7$. We aim to find a monochromatic C_n in this coloring. Let $A \cup B \cup C \cup D$ be a partition of $V := V(G)$ satisfying either conditions (a), (b) of $EC_1(\alpha, \delta)$ or (a)-(c) of $EC_3(\mu, c_1, c_2, \delta)$.

Now we find large subsets $A_2 \subset A$, $B_2 \subset B$, $C_2 \subset C$, $D_2 \subset D$ such that the induced coloring of the graph $G_2 := G[A_2 \cup B_2 \cup C_2 \cup D_2]$ is either EC_1 (α)-complete or EC_3 (μ, c_1, c_2)-complete. To this goal, we first remove from A all the vertices with low degrees to B, C or D (in an appropriate color). More precisely, a vertex $v \in A$ has *low* degree if either
$$\deg_R(v, B) < (1 - \delta^{1/2})|B| \text{ or } \deg_B(v, C) < (1 - \delta^{1/2})|C| \text{ or } \deg_G(v, D) < (1 - \delta^{1/2})|D|.$$

From the condition (b) in $EC_1\,(\alpha,\delta)$ or (c) in $EC_3\,(\mu,c_1,c_2,\delta)$ it follows that the number of these low degree vertices in A is at most $6\delta^{1/2}|A|$. Analogously, we define (and estimate the number of) the low degree vertices for the sets B, C and D. Let E_1 be the set of all the low degree vertices. Then

$$|E_1| \le 6\delta^{1/2}(|A| + |B| + |C| + |D|) \le 24\delta^{1/2}n.$$

Let $A_1 = A \setminus E_1$, $B_1 = B \setminus E_1$, $C_1 = C \setminus E_1$ and $D_1 = D \setminus E_1$. We observe that every vertex $v \in A_1$ is adjacent to at least

$$(1 - \delta^{1/2})|B| - 6\delta^{1/2}|B| \ge (1 - 7\delta^{1/2})|B_1| \ge (1 - \delta^{1/3})|B_1|$$

vertices in B_1 by red edges. Similarly, we have

$$\deg_B(v, C_1) \ge (1 - \delta^{1/3})|C_1| \text{ and } \deg_G(v, D_1) \ge (1 - \delta^{1/3})|D_1|.$$

We get similar inequalities for the sets B_1, C_1 and D_1 and for appropriate colors.

Then Lemma 15 implies that for any $a \in A_1$ and $b \in B_1$, the bipartite graph $G^R[A_1, B_1]$ contains a red (a,b)-path of any odd length between 3 and $2(1-2\delta^{1/3})\min\{|A_1|, |B_1|\} - 1$, and that for any two vertices $a_1, a_2 \in A$ it also contains a red (a_1, a_2)-path of any even length between 2 and $2(1-2\delta^{1/3})\min\{|A_1|, |B_1|\}-2$. The same holds for the other pairs of sets and the corresponding colors.

In particular, this means that there are no red vertex-disjoint edges between A_1 and C_1: if a_1c_1 and a_2c_2 were two such edges, then for any even number k satisfying

$$4 \le k \le 2(1 - 2\delta^{1/3})(\min\{|A_1|, |B_1|\} + \min\{|C_1|, |D_1|\}) - 4, \tag{4}$$

we can find an (a_1, a_2)-path P in $G^R[A_1, B_1]$ and a (c_1, c_2)-path Q in $G^R[C_1, D_1]$ such that $e(P) + e(Q) = k$. Clearly, $P \cup Q \cup \{a_1c_1, a_2c_2\}$ is a copy of C_{k+2}. At this point we must distinguish whether the original coloring of G was of type $EC_1\,(\alpha,\delta)$ or $EC_3\,(\mu, c_1, c_2, \delta)$.

The first case is easy: since in any $EC_1\,(\alpha,\delta)$-type coloring of G the sizes of the sets A, B, C, D are at least $(1-\alpha)2n/4$, we have that

$$|A_1|, |B_1|, |C_1|, |D_1| \ge \frac{(1-\alpha)n}{2} - 24\delta^{1/2}n.$$

Consequently, k can be any even number between 4 and $2(1-2\delta^{1/3})(1-\alpha-48\delta^{1/2})n - 4$. Since $\delta \le \alpha \le 10^{-12}$, it is easy to see that

$$2(1-2\delta^{1/3})(1-\alpha-48\delta^{1/2})n \ge 2(1-2\alpha^{1/3})(1-49\alpha^{1/2})n \ge (1-51\alpha^{1/3})(2n) \ge n+2. \tag{5}$$

In the second case, the condition (a) of $EC_3\,(\mu, c_1, c_2, \delta)$ implies that

$$|A_1|, |B_1|, |C_1| \ge \frac{(1-c_1\mu)n}{2} - 24\delta^{1/2}n$$

and

$$|D_1| \ge \mu\frac{2n}{4} - 24\delta^{1/2}n.$$

Since $\delta \le \delta_7 \le (1-c_1)^3\mu^3/10^6$, k can be as large as

$$2(1-2\delta^{1/3})\left(\frac{(1-c_1\mu)n}{2} + \frac{\mu n}{2} - 48\delta^{1/2}n\right) > (1 + (1-c_1)\mu - 100\delta^{1/3})n \ge n+2. \tag{6}$$

In either case, we can take $k = n - 2$ above and find a red copy of $C_{k+2} = C_n$ in G. This means that there are no red edges in $E(A_1, C_1)$ with the exception of at most one red star. By the same argument in which we use the green bipartite graphs $G^G[A_1, D_1]$ and $G^G[C_1, B_1]$ instead of $G^R[A_1, B_1]$ and $G^R[C_1, D_1]$, there are no green edges in $E(A_1, C_1)$ with the exception of at most one green star.

Similar arguments show that there is at most one red and one green star in $E(B_1, D_1)$, one red and one blue star in $E(A_1, D_1)$ and in $E(B_1, C_1)$, and one green and one blue star in $E(A_1, B_1)$ and in $E(C_1, D_1)$. We remove the centers of these (at most 12) stars from A_1, B_1, C_1, D_1 and obtain sets A_2, B_2, C_2, D_2. Let $E_2 = V(G) \setminus (A_2 \cup B_2 \cup C_2 \cup D_2)$.

Clearly, the induced coloring of the graph $G_2 = G[A_2 \cup B_2 \cup C_2 \cup D_2]$ is either $EC_1\,(\alpha)$-complete or $EC_3\,(\mu, c_1, c_2)$-complete. Although these are very nice colorings of G_2, they may not have any monochromatic C_n. Therefore we still need to use the vertices in E_2 to build our monochromatic C_n in G.

Definition 19. *For a vertex $v \in E_2$, we say that*

v **is $R1$-type** *if either* $\deg_R(v, A_2)$, $\deg_R(v, C_2) \geq 4$ *or* $\deg_R(v, B_2)$, $\deg_R(v, D_2) \geq 4$*;*

v **is $R2$-type** *if either* $\deg_R(v, A_2)$, $\deg_R(v, D_2) \geq 4$ *or* $\deg_R(v, B_2)$, $\deg_R(v, C_2) \geq 4$*;*

v **is $B1$-type** *if either* $\deg_B(v, A_2)$, $\deg_B(v, B_2) \geq 4$ *or* $\deg_B(v, C_2), \deg_B(v, D_2) \geq 4$*;*

v **is $B2$-type** *if either* $\deg_B(v, A_2)$, $\deg_B(v, D_2) \geq 4$ *or* $\deg_B(v, B_2), \deg_B(v, C_2) \geq 4$*;*

v **is $G1$-type** *if either* $\deg_G(v, A_2)$, $\deg_G(v, B_2) \geq 4$ *or* $\deg_G(v, C_2), \deg_G(v, D_2) \geq 4$*;*

v **is $G2$-type** *if either* $\deg_G(v, A_2)$, $\deg_G(v, C_2) \geq 4$ *or* $\deg_G(v, B_2), \deg_G(v, D_2) \geq 4$.

The next claim shows that E_2 contains only a few vertices of the types defined above.

Claim 20. *Either there exists a monochromatic C_n in G or there is at most one vertex of each of the above types.*

Proof. Suppose that G contains two vertices v_1 and v_2 of type $R1$. We will show that G has a red copy of C_n. Assume, without loss of generality, that $\deg_R(v_1, A_2) \geq 4$ and $\deg_R(v_1, C_2) \geq 4$ and let $a_1 \in A_2$, $c_1 \in C_2$ be any red neighbors of v_1.

Now if $\deg_R(v_2, A_2) \geq 4$ and $\deg_R(v_2, C_2) \geq 4$, then there are red neighbors $a_2 \in A_2, c_2 \in C_2$ of v_2 that are distinct from a_1, c_1. It follows from Lemma 15 that for any even number k satisfying

$$4 \leq k \leq 2(1 - 2\delta^{1/3})\,(\min\{|A_2|, |B_2|\} + \min\{|C_2|, |D_2|\}) - 4, \tag{7}$$

there exist an even red (a_1, a_2)-path P in $G^R[A_2, B_2]$ and an even red (c_1, c_2)-path Q in $G^R[C_2, D_2]$ such that $e(P) + e(Q) = k$. Clearly, $P \cup Q \cup \{v_1a_1, a_2v_2, v_2c_2, c_1v_1\}$ form a red copy of C_{k+4}. The same type of analysis that we have done in (5) and (6) shows that we can take $k = n - 4$ and find a red copy of C_n in G.

If $\deg_R(v_2, B_2) \geq 4$ and $\deg_R(v_2, D_2) \geq 4$, then we proceed similarly: we take any red neighbors $b_2 \in B_2$ and $d_2 \in D_2$ of v_2 and find red paths P from a_1 to b_2 with edges in $G^R[A_2, B_2]$ and Q from c_1 to d_2 with edges in $G^R[C_2, D_2]$ such that $P \cup Q \cup \{v_1a_1, b_2v_2, v_2d_2, c_1v_1\}$ is a red cycle of length n.

By symmetry, if G has two vertices of type $R2$, $B1$, $B2$, $G1$ or $G2$, then we can also find a monochromatic C_n. We omit the details here. □

Remark 21. *In order to prove Claim 20, it suffices to say that a vertex v is of type $R1$ if either* $\deg_R(v, A_2)$*,* $\deg_R(v, C_2) \geq 2$ *or* $\deg_R(v, B_2)$*,* $\deg_R(v, D_2) \geq 2$*. Only later we will need the stronger definition of $R1$-type vertices.*

We say that a vertex $v \in E_2$ is of **type $R*$** (or **$R*$-type**) if it is either $R1$-type or $R2$-type. We define $B*$-type and $G*$-type vertices similarly. Notice, for example, that any vertex $v \in E_2$ that satisfies $\deg_R(v, A_2 \cup B_2) \geq 7$ and $\deg_R(v, C_2 \cup D_2) \geq 7$ must be of type $R*$.

Denote by F be the set of vertices of type $R*$, $G*$ or $B*$. By Claim 20, we have that $|F| \leq 6$. Let $E_2' = E_2 \setminus F$. We define a partition $A_2' \cup B_2' \cup C_2' \cup D_2'$ of E_2' as follows: We put a vertex $v \in E_2'$

to A_2' if $\deg_R(v, B_2) \geq |B_2| - 12$, $\deg_B(v, C_2) \geq |C_2| - 12$, and $\deg_G(v, D_2) \geq |D_2| - 12$;

to B_2' if $\deg_R(v, A_2) \geq |A_2| - 12$, $\deg_G(v, C_2) \geq |C_2| - 12$, and $\deg_B(v, D_2) \geq |D_2| - 12$;

to C_2' if $\deg_B(v, A_2) \geq |A_2| - 12$, $\deg_G(v, B_2) \geq |B_2| - 12$, and $\deg_R(v, D_2) \geq |D_2| - 12$;

to D_2' if $\deg_G(v, A_2) \geq |A_2| - 12$, $\deg_B(v, B_2) \geq |B_2| - 12$, and $\deg_R(v, C_2) \geq |C_2| - 12$.

We decide ties arbitrarily so that A_2', B_2', C_2', D_2' are pairwise disjoint.

Is this really a partition of E_2'? Indeed, for each vertex $v \in E_2'$, since v is not of $R*$-type, we must have either $\deg_R(v, A_2 \cup B_2) \leq 6$ or $\deg_R(v, C_2 \cup D_2) \leq 6$ not to contradict our observation above. Furthermore, we must also have that either $\deg_B(v, A_2 \cup C_2) \leq 6$ or $\deg_B(v, B_2 \cup D_2) \leq 6$, and either $\deg_G(v, A_2 \cup D_2) \leq 6$ or $\deg_G(v, B_2 \cup C_2) \leq 6$. Without loss of generality assume that $\deg_R(v, C_2 \cup D_2) \leq 6$ and $\deg_B(v, B_2 \cup D_2) \leq 6$. Then $\deg_G(v, D_2) \geq |D_2| - 12$, which implies $\deg_G(v, B_2 \cup C_2) \leq 6$. From this we conclude that $\deg_B(v, C_2) \geq |C_2| - 12$ and $\deg_R(v, B_2) \geq |B_2| - 12$. Hence, v belongs to A_2'. The other 3 possibilities yield that v is in one of B_2', C_2' or D_2'.

We put $\tilde{A_2} := A_2 \cup A_2'$, $\tilde{B_2} := B_2 \cup B_2'$, $\tilde{C_2} := C_2 \cup C_2'$ and $\tilde{D_2} := D_2 \cup D_2'$, therefore, $\tilde{A_2} \cup \tilde{B_2} \cup \tilde{C_2} \cup \tilde{D_2} = A_2 \cup B_2 \cup C_2 \cup D_2 \cup E_2' = V(G) \setminus F$ and

$$|\tilde{A_2} \cup \tilde{B_2} \cup \tilde{C_2} \cup \tilde{D_2}| = 2n - |F| \geq 2n - 6.$$

Recall that $|E_2'| \leq |E_2| \leq |E_1| + 12 \leq 25\delta^{1/2}n$. If the original coloring of G was of type $EC_3\,(\mu, c_1, c_2, \delta)$, then $\tilde{A_2}$ must be the largest among $\tilde{A_2}$, $\tilde{B_2}$, $\tilde{C_2}$, $\tilde{D_2}$ because condition (b) of $EC_3\,(\mu, c_1, c_2, \delta)$ holds for the original partition of G and $25\delta^{1/2} \leq 25\delta_7^{1/2} \leq \mu$. If it was $EC_1\,(\alpha, \delta)$, then we may assume the same by symmetry.

Notice that in either case, $|\tilde{A_2}| \geq n/2 - 1$. If two of the sets $\tilde{A_2}$, $\tilde{B_2}$, $\tilde{C_2}$ and $\tilde{D_2}$, say $\tilde{A_2}$ and $\tilde{B_2}$, have at least $n/2$ vertices, then there is a monochromatic red C_n in $G^R[\tilde{A_2}, \tilde{B_2}]$ by Lemma 16 and Remark 18 (applied with $\beta = 50\delta^{1/2}$).

Thus, we may assume that $|\tilde{B_2}| = n/2 - b$, $|\tilde{C_2}| = n/2 - c$ and $|\tilde{D_2}| = n/2 - d$, where $b, c, d > 0$. We put $|\tilde{A_2}| = n/2 + a$, where $a \geq -1$, and $|F| = f \geq 0$. Then

$$a + f = b + c + d. \tag{8}$$

We distinguish two cases: $a \geq 0$ and $a = -1$.

Case 1: $a \geq 0$. We first prove that $G[A_2]$ contains a long monochromatic path.

Claim 22. *The graph $G[A_2]$ contains either a red path P_{2b+1}, or a blue path P_{2c+1}, or a green path P_{2d+1}.*

Proof. Suppose that none of those paths exists. By Lemma 14, $G[A_2]$ has at most $(2b-1)|A_2|/2$ red edges, $(2c-1)|A_2|/2$ blue edges and $(2d-1)|A_2|/2$ green edges. Therefore,

$$\left(\frac{|A_2|-1}{2}\right)|A_2| \le \left(\frac{2b-1}{2}+\frac{2c-1}{2}+\frac{2d-1}{2}\right)|A_2|.$$

From $|\tilde{A_2}| = |A_2| + |A_2'|$ and $|A_2'| \le |E_2'| \le 25\delta^{1/2}n$, it follows that

$$\frac{n}{2} + a - 25\delta^{1/2}n - 1 \le |\tilde{A_2}| - |E_2'| - 1 = |A_2| - 1 \le 2(b+c+d) - 3 \overset{(8)}{=} 2(a+f) - 3.$$

Hence, we have that $a \ge n/2 - 25\delta^{1/2}n - 2f + 2$. From this we derive a lower bound on the order of the set A from the original partition of $V(G)$, namely

$$|A| \ge |A_2| \ge |\tilde{A_2}| - 25\delta^{1/2}n \ge n - 50\delta^{1/2}n - 2f + 2.$$

But this is a contradiction: in $EC_1(\alpha,\delta)$, from condition (a) and from $\delta \le \alpha \le 10^{-12}$, we have

$$|A| = (2n) - |B| + |C| + |D| \le 2n - 3(1-\alpha)\frac{2n}{4} = (1+3\alpha)\frac{n}{2} < \frac{3n}{4} < n - 50\delta^{1/2}n - 2f + 2;$$

in $EC_3(\mu, c_1, c_2, \delta)$, the bounds for $|A \cup D|$ and $|D|$ given in (a) and (b) imply that

$$|A| = |A \cup D| - |D| \le (1 + (c_2 - 0.5)\mu)n < n - 50\delta^{1/2}n - 2f + 2,$$

because $c_2 < 0.5$ and $\delta \le \delta_7 \le ((0.5 - c_2)\mu/100)^2$. □

Assume there exists a green path P_{2d+1} in $G[A_2]$.

If $d \le a$, then we find a green C_n using Lemma 17 applied with $H = G^G$, $\tilde{X} = \tilde{A_2}$, $X = A_2$, $X' = A_2'$, $\tilde{Y} = \tilde{D_2}$, $Y = D_2$, $Y' = D_2'$, $\beta = 1/100$ and $t = d$. To verify its assumptions, it suffices to recall that $|A_2'|, |D_2'| \le 25\delta^{1/2}n$ and, moreover, to notice that if the original coloring was $EC_1(\alpha,\delta)$, then we have that $d \le (\alpha + 50\delta^{1/2})n/2$, otherwise, the coloring was $EC_3(\mu, c_1, c_2, \delta)$ and we have $d \le (1-\mu)n/2$.

If $d > a$, then for every vertex v in F we find a color $q \in \{R, B, G\}$ for which

$$\deg_q(v, A_2 \setminus P) \ge \frac{|A_2 \setminus P|}{3}. \tag{9}$$

Suppose there are $d-a$ vertices for which this color is green. We add theses vertices to $D_2' \subset \tilde{D_2}$, so that, after adding these vertices, $|\tilde{D_2}| = n/2 - d + (d-a) = n/2 - a$, and we use Lemma 17 again to find a green C_n.

Otherwise, there exist at most $d-a-1$ vertices in F such that $\deg_G(v, A_2 \setminus P) \ge |A_2 \setminus P|/3$. If there are b vertices for which the color q in (9) is red, then we add theses vertices to $B_2' \subset \tilde{B_2}$ so that, after adding these vertices, $|\tilde{B_2}| \ge n/2$. Applying Lemma 16 yields a red cycle C_n in $G^R[\tilde{A_2}, \tilde{B_2}]$.

Otherwise, there are at most $b-1$ vertices in F such that the color q in (9) is red. By the same argument, there either exists a blue C_n in $G^B[\tilde{A_2}, \tilde{C_2}]$ or at most $c-1$ vertices in F satisfies (9) with $q = B$. But the latter is impossible because then

$$|F| \le (d-a-1) + (b-1) + (c-1) = |F| - 3.$$

Other cases, when there is a blue path P_{2c+1} or a red path P_{2b+1} in $G[A_2]$, are handled similarly.

Case 2: $a = -1$. In this case, we have that $n/2 - 1 = |\tilde{A_2}| \geq |\tilde{B_2}|, |\tilde{C_2}|, |\tilde{D_2}|$, $|F| \leq 6$, and $|\tilde{A_2}| + |\tilde{B_2}| + |\tilde{C_2}| + |\tilde{D_2}| + |F| = 2n$, therefore, $|\tilde{B_2}|, |\tilde{C_2}|, |\tilde{D_2}| \geq n/2 - 3$ and $|F| \geq 4$.

Let us first solve the easier sub-case when $|F| = 4$. It follows form (8) that $b = c = d = 1$, hence

$$|\tilde{A_2}| = |\tilde{B_2}| = |\tilde{C_2}| = |\tilde{D_2}| = \frac{n}{2} - 1.$$

Recall that all the vertices of F are either $R*$-, $G*$- or $B*$-type. As $|F| = 4$, two of them, say u and v, must have the same color type. Since the sizes of $\tilde{A}_2, \tilde{B}_2, \tilde{C}_2, \tilde{D}_2$ are all equal and our coloring is symmetric, without loss of generality, we may assume that u and v are $R*$-type, say u is of $R1$-type and v is of $R2$-type. We may also assume that $\deg_R(u, A_2)$, $\deg_R(u, C_2) \geq 4$ and $\deg_R(v, A_2)$, $\deg_R(v, D_2) \geq 4$. Then we add u to the set D_2' and v to C_2' and find a red C_n in $G[\tilde{C}_2, \tilde{D}_2]$ using Lemma 16.

When $|F| = 5$ or $|F| = 6$, we will need to look at the edges inside the sets A_2,B_2,C_2 and D_2, and use the following claim.

Claim 23. *If there are two vertices of type $R*$ ($B*$, $G*$, respectively) and at least one red (blue, green, respectively) edge inside any of the sets A_2, B_2, C_2, D_2 then we can find a red (blue, green, respectively) C_n.*

Proof. Suppose that there exist two vertices of type $R*$, say u of type $R1$ and v of type $R2$. By symmetry, we may assume that $\deg_R(u, A_2)$, $\deg_R(u, C_2) \geq 4$ and $\deg_R(v, A_2)$, $\deg_R(v, D_2) \geq 4$. Let xy be any red edge in A_2. We choose distinct red neighbors $u_A \in A_2 \setminus \{x, y\}$, $u_C \in C_2$ of u, and $v_A \in A_2 \setminus \{x, y\}$, $v_D \in D_2$ of v.

Since the coloring induced by $A_2 \cup B_2 \cup C_2 \cup D_2$ is $EC_1(\alpha)$-complete, there is

- a (u_A, x)-path P_1 of length 2 in $G^R[A_2, B_2]$;
- a (v_A, y)-path P_2 of any even length between 2 and $2\min\{|A_2| - 3, |B_2| - 1\}$ in $G^R[A_2 \setminus V(P_1), B_2 \setminus V(P_1)]$;
- a (u_C, v_D)-path P_3 of any odd length between 1 and $2\min\{|C_2|, |D_2|\} - 1$ in $G^R[C_2, D_2]$.

Hence, $P_1 \cup P_2 \cup P_3 \cup \{uu_A, uu_C, vv_A, vv_D, xy\}$ is a red cycle of any even length between 10 and

$$2\min\{|A_2|, |B_2|\} + 2\min\{|C_2|, |D_2|\} - 2 > n.$$

In particular, we can find a red C_n. The cases when xy is in B_2, C_2 or D_2 are handled similarly. □

Therefore notice that we cannot have $|F| = 6$. Indeed, as there are at most two special vertices in each color, the only way to have $|F| = 6$ is if we have two vertices of type $R*$, two of type $B*$ and two of type $G*$. Therefore, applying the above claim to each color, there is no way to color any edge inside A_2, B_2, C_2 and D_2 without creating a C_n.

The last remaining case is $|F| = 5$. Without loss of generality we may assume that $|\tilde{D_2}| \leq |\tilde{B_2}|, |\tilde{C_2}|$, and hence

$$(|\tilde{A_2}|, |\tilde{B_2}|, |\tilde{C_2}|, |\tilde{D_2}|) = \left(\frac{n}{2} - 1, \frac{n}{2} - 1, \frac{n}{2} - 1, \frac{n}{2} - 2\right).$$

By permuting the names of the colors (if needed), we may assume that F contains two vertices r_1, r_2 of type $R*$, two vertices b_1, b_2 of type $B*$ and a vertex g of type $G*$. Using the Claim 23, we may assume that all the edges within A_2, B_2, C_2, D_2 are green.

We construct an auxiliary graph H with vertex set $\{r_1, r_2, b_1, b_2, A_2, B_2, C_2, D_2\}$. For all vertices $u, v \in V(H)$, we put $\{u, v\}$ as an edge of H, except when $\{u, v\} \subset \{r_1, r_2, b_1, b_2\}$. We 3-multicolor the edges of H in the following way: edges $\{U, V\}$ with $U, V \in \{A_2, B_2, C_2, D_2\}$ receive only one color: the unique color that appears in $G[U, V]$; edges $\{u, V\}$ with $u \in \{r_1, r_2, b_1, b_2\}$ and $V \in \{A_2, B_2, C_2, D_2\}$ receive color red (blue, green, respectively) if $\deg_R(u, V) \geq 4$ ($\deg_B(u, V) \geq 4$, $\deg_G(u, V) \geq 4$, respectively).

To finish the proof we need to treat 16 possibilities, according to how the vertices r_1, r_2, b_1, b_2 to connect A_2, B_2, C_2, D_2 in Definition 19. Although these possibilities are not symmetric, they can be represented by one of the following four drawing (see Fig. 3), in which (X, Y, Z, W) is some suitable permutation of (A_2, B_2, C_2, D_2).

Indeed, one can always draw H in a way that the red edges between $\tilde{A_2}, \tilde{B_2}, \tilde{C_2}, \tilde{D_2}$ are drawn horizontally, the blue edges vertically and the green edges diagonally. After this, we can perform horizontal and/or vertical reflections to position the vertices r_1, r_2 like in the Fig. 3. We then look at the relative position of vertices b_1 and b_2. This gives us only four sub-cases to treat.

We will use $\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{W}$ to denote $\tilde{A_2}, \tilde{B_2}, \tilde{C_2}, \tilde{D_2}$ and X', Y', Z', W' to denote A_2', B_2', C_2', D_2' in the correspondent order. For each of the four above drawings, we need to treat four possibilities, according to which of the sets $\tilde{X}$, $\tilde{Y}$, $\tilde{Z}$, $\tilde{W}$ has order $n/2 - 2$.

If in any drawing below $|\tilde{X}| = n/2 - 2$ or $|\tilde{Y}| = n/2 - 2$, then we could add r_1 to W' and r_2 to Z' and find a red C_n in $G[\tilde{Z}, \tilde{W}]$ using Lemma 16.

In sub-cases $2a$ and $2c$, if $|\tilde{W}| = n/2-2$, then we could add b_1 to X' and b_2 to Z' and find a blue C_n in $G[\tilde{X}, \tilde{Z}]$, also using Lemma 16. In a similar way, in sub-cases $2b$ and $2d$, if $|\tilde{Z}| = n/2 - 2$, then we could find a blue C_n in $G[\tilde{Y}, \tilde{W}]$.

The next 2 possibilities require a little more work.

Sub-case 2a and $|\tilde{Z}| = n/2 - 2$: If the edge $\{b_2, W\}$ was blue, then the vertex b_2 would be of type $B1$. But there cannot be two vertices of type $B1$ in F by Claim 20. If the edge $\{b_2, W\}$ was red, then we could add b_2 and r_2 to the set Z' and r_1 to the set W and find a red C_n in $G[\tilde{Z}, \tilde{W}]$ by Lemma 16. Therefore, $\{b_2, W\}$ must be green-exclusive. A similar argument shows that $\{b_1, W\}$ must also be green-exclusive. Hence,

$$\deg_G(b_1, W), \deg_G(b_2, W) \geq |W| - 6.$$

But now we can add b_1 and b_2 to the set X' and obtain that $|\tilde{X}| = n/2 + 1$ and $|\tilde{W}| = n/2 - 1$. Since all the edges within X are green, we find a path of length 2 in X and extend this path to a cycle of length n in $G[\tilde{X}, \tilde{W}]$ using Lemma 17.

Sub-case 2b and $|\tilde{W}| = n/2 - 2$: Here the edge $\{b_2, Y\}$ cannot be blue (otherwise we would have two vertices of type $B1$) and cannot be red (otherwise we could add b_2 to X' and r_2 to Y' and find a red C_n by Lemma 16). Therefore, $\{b_2, Y\}$ must be green-exclusive. Similarly, the edge $\{r_2, Z\}$ cannot be red (otherwise we would have two vertices of type $R1$) neither blue (otherwise we could add b_2 to Z' and r_2 to X' and find a blue C_n in $G[\tilde{X}, \tilde{Z}]$ using Lemma 16). Therefore, $\{b_2, Y\}$ must be green-exclusive. Then, however, we can add r_2 to Y and b_2 to Z' and find a green C_n in $G[\tilde{Y}, \tilde{Z}]$ using Lemma 16.

The last two possibilities are very similar to the previous two, so we only indicate the edges that we must look at.

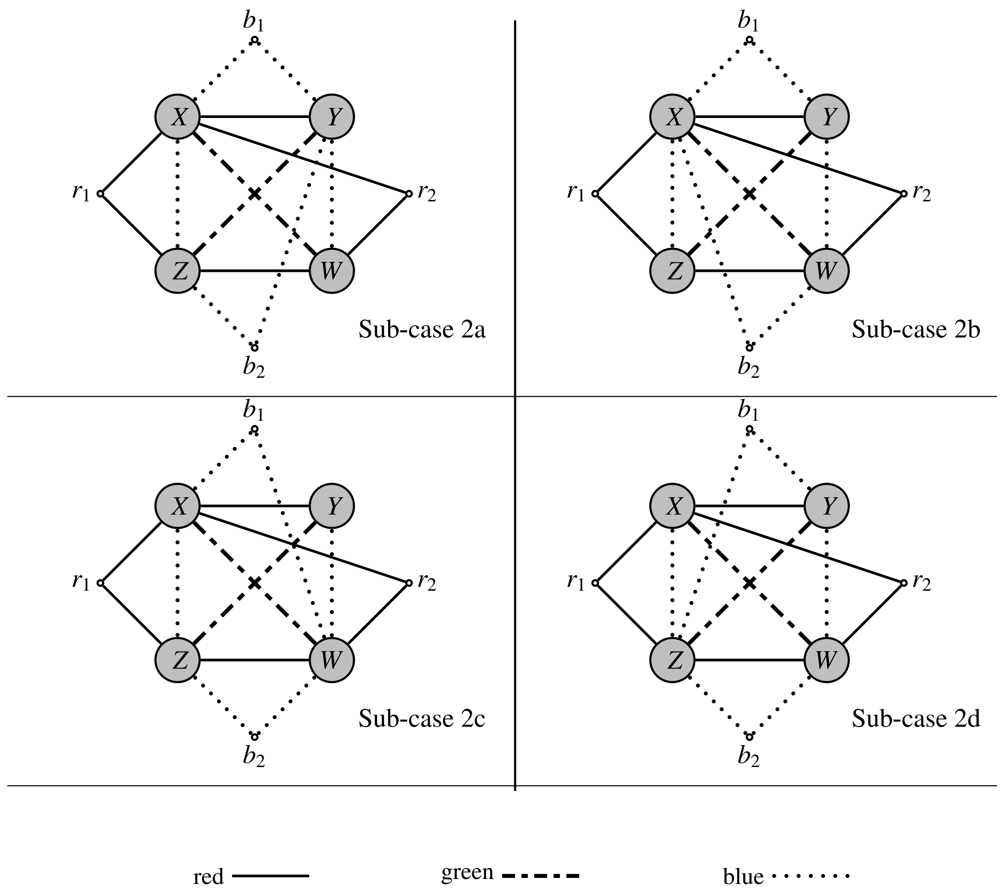

Figure 4: Possibilities for H

Sub-case 2c and $|\tilde{Z}| = n/2 - 2$: First, observe that the edges $\{b_2, Y\}$ and $\{r_1, Y\}$ must be both green-exclusive. Furthermore, $\{b_1, Y\}$ must be green-exclusive as well. Then we add b_1, b_2 and r_1 to the set Z', find a path of length 2 in Z and use Lemma 17 to find a green C_n in $G[\tilde{Y}, \tilde{Z}]$.

Sub-case 2d and $|\tilde{W}| = n/2 - 2$: Here, both the edges $\{b_2, Y\}$ and $\{r_1, Y\}$ must be green-exclusive. Then, after we add b_2 and r_1 to the set Z', find a path of length 2 in Z and we apply Lemma 17 to find a green C_n in $G[\tilde{Y}, \tilde{Z}]$. □

8. Proof of Lemma 6

We set

$$\alpha_6 = 10^{-18}$$

and consider any $\alpha \leq \alpha_6$. Note that, for every $\delta \leq \alpha$, any coloring of type $EC_2(\alpha, \delta)$ is also of type $EC_2(\alpha, \alpha)$, hence, we may assume that $\delta = \alpha$. Take

$$n_6 = \lceil \alpha^{-6} \rceil.$$

Consider any 3-coloring of $G = K_{2n}$ of type $EC_2(\alpha, \alpha)$ and let $A \cup B \cup C \cup D$ be a partition of $V := V(G)$ satisfying both conditions (a), (b) of $EC_2(\alpha, \alpha)$. Similarly to the proof of Lemmas 5 and 7, a vertex $v \in A$ has "low" degree if

$$\deg_R(v, B) < (1 - \alpha^{1/2})|B|, \quad \deg_G(v, C) < (1 - \alpha^{1/2})|C|, \quad \text{or} \quad \deg_B(v, D) < (1 - \alpha^{1/2})|D|;$$

a vertex $v \in B$ has "low" degree if

$$\deg_R(v, A) < (1 - \alpha^{1/2})|A|, \quad \deg_G(v, C) < (1 - \alpha^{1/2})|C|, \quad \text{or} \quad \deg_B(v, D) < (1 - \alpha^{1/2})|D|;$$

a vertex $v \in C$ has "low" degree if

$$\deg_G(v, A \cup B) < (1 - \alpha^{1/2})(|A| + |B|),$$

and, finally, a vertex $v \in D$ has "low" degree if

$$\deg_B(v, A \cup B) < (1 - \alpha^{1/2})(|A| + |B|).$$

By the condition (b) in $EC_2(\alpha, \alpha)$, it follows that the number of low degree vertices in A (B, C, D, respectively) is at most $6\alpha^{1/2}|A|$ ($6\alpha^{1/2}|B|$, $\alpha^{1/2}|C|$, $\alpha^{1/2}|D|$, respectively). Together, there are at most $28\alpha^{1/2}n$ low degree vertices that we put into a new set say E_1. We define $A_1 := A \setminus E_1$, $B_1 := B \setminus E_1$, $C_1 := C \setminus E_1$, $D_1 := D \setminus E_1$.

Thus, all the vertices in $A_1 \cup B_1$ are adjacent to at least $(1-2\alpha^{1/2})|C| \geq (1-2\alpha^{1/2})|C_1|$ vertices in C_1 by green edges and all the vertices in C_1 are adjacent to at least $(1 - 7\alpha^{1/2})(|A| + |B|) \geq (1 - 7\alpha^{1/2})(|A_1| + |B_1|)$ by green edges. A similar statements hold for other colors and sets. If $|C_1| \geq n/2$, then we greedily find a monochromatic C_n in $[C_1, A_1 \cup B_1]$ because every two vertices of C_1 have $(1 - 14\alpha^{1/2})(|A_1| + |B_1|)$ common neighbors and

$$|A_1| + |B_1| \geq 2(1 - \alpha)\frac{2n}{4} - 28\alpha^{1/2}n > \frac{n}{2(1 - 14\alpha^{1/2})}.$$

By the same type of argument we are also done if $|D_1| \geq n/2$. Hence suppose that $|C_1| = n/2 - c$ and $|D_1| = n/2 - d$ for some $c, d > 0$. From $|E_|| \leq 28\alpha^{1/2}n$ and condition (a) in $EC_2(\alpha, \alpha)$ it follows that $c, d \leq 29\alpha^{1/2}n$.

We claim that there is neither a green path in $A_1 \cup B_1$ of length $2c$ nor a blue path in $A_1 \cup B_1$ of length $2d$. To the contrary, suppose that P is such a green path in $A_1 \cup B_1$, with length $2c$ and endpoints a_1 and a_2.

Since all the vertices in $A_1 \cup B_1$ have high green degrees to C_1, we find $c_1 \neq c_2 \in C_1$ such that a_1c_1, a_2c_2 are green. Now, every two vertices in C_1 have at least

$$(1 - 14\alpha^{1/2})(|A_1| + |B_1|) - 2c \geq 2(1 - \alpha)\frac{2n}{4} - 28\alpha^{1/2}n - 2 \cdot 29\alpha^{1/2}n \geq \frac{n}{2} > |C_1|$$

common green neighbors in $A_1 \cup B_1 \setminus V(P)$. Hence, we greedily find a green (c_1, c_2)-path P' that avoids $V(P)$ and saturates all the vertices of C_1. Then $P \cup P'$ is a green monochromatic C_n.

Consequently, $A_1 \cup B_1$ contains at most

$$(2c-1)\frac{|A_1|+|B_1|}{2} \le 29\alpha^{1/2}n\left(2n-2\left(\frac{n}{2}-29\alpha^{1/2}n\right)\right) \le 58\alpha^{1/2}n^2$$

green edges and at most $(2d-1)(|A_1|+|B_1|)/2 \le 58\alpha^{1/2}n^2$ blue edges. We again remove from A_1 and B_1 all the vertices adjacent to more than $\alpha^{1/6}n$ vertices of $A_1 \cup B_1$ by green or blue edges and put them into E_1. Call the new sets A_2, B_2, C_2, D_2 and E_2 (we set $C_2 := C_1$ and $D_2 := D_1$). Note that we removed at most $\alpha^{1/6}n$ vertices, and so $|E_2| \le 2\alpha^{1/6}n$. Therefore, $G^R[A_2 \cup B_2]$ has minimum degree $|A_2 \cup B_2| - 3\alpha^{1/6}n$.

If $|A_2 \cup B_2| \ge n$, then $|A_2 \cup B_2| - 3\alpha^{1/6}n > |A_2 \cup B_2|/2$. Hence, $G^R[A_2 \cup B_2]$ contains C_n by Theorem 13.

Otherwise, $|A_2 \cup B_2| < n$ and $n_{AB} := n - |A_2 \cup B_2| \ge 1$. We also notice that $n/2 - |C_2| = n/2 - |C_1| = c$ and $n/2 - |D_2| = n/2 - |D_1| = d$. Suppose there is $c \le 29\alpha^{1/2}n$ vertices $e_1, \dots, e_c$ in E_2, each with at least $2c$ green neighbors in $A_2 \cup B_2$. For each e_i, take two of its green neighbors $a_i, b_i \in A_2 \cup B_2$. Since e_i has $2c$ such neighbors, we can select all a_i, b_i distinct.

Next, we find a green neighbor $c_1 \in C_2$ of a_1, a green neighbor $c_{k+1} \in C_2$ of b_k, and for all b_{i-1}, a_i, where $i = 2, \dots, k$, a common green neighbor $c_i \in C_2$. Again, all c_i's may be chosen distinct because any two vertices in $A_2 \cup B_2$ have at least

$$(1-4\alpha^{1/2})|C_1| = (1-4\alpha^{1/2})|C_2| \ge (1-4\alpha^{1/2})\left(\frac{n}{2} - 29\alpha^{1/2}n\right) > 58\alpha^{1/2}n > 2c$$

common green neighbors in C_2. Finally, we may greedily find a green (c_1, c_{k+1})-path avoiding all a_i, b_i, c_i's and saturating the remaining vertices of C_2, because all the pairs of vertices in C_2 have a large common neighborhood to $A_2 \cup B_2$:

$$\begin{aligned}(1-14\alpha^{1/2})(|A_1|+|B_1|) - \alpha^{1/6}n &\ge 2(1-\alpha)\frac{2n}{4} - 28\alpha^{1/2}n - \alpha^{1/6}n \\ &\ge \frac{n}{2} + 58\alpha^{1/2}n > |C_2| + 2c.\end{aligned}$$

Hence, we only need to settle the case in which there are less than c vertices in E_2 that have at least $2c$ green neighbors in $A_2 \cup B_2$. In the same way, we may also assume that less than d vertices in E_2, have at least $2d$ blue neighbors in $A_2 \cup B_2$.

Thus, there are at least $2n - |A_2 \cup B_2| - |C_2| - |D_2| - (c-1) - (d-1) > n - |A_2 \cup B_2| = n_{AB}$ vertices in E_2, each with at least $|A_2 \cup B_2| - 2(c+d) > 2|A_2 \cup B_2|/3$ red neighbors in $A_2 \cup B_2$. Let F be a set with any n_{AB} of these vertices. Since $n_{AB} = |F| \le |E_2| \le 2\alpha^{1/6}n$, the graph $G^R[A_2 \cup B_2 \cup F]$ has the minimum degree at least $2|A_2 \cup B_2|/3 > |A_2 \cup B_2 \cup F|/2$. Since $|A_2 \cup B_2| + |F| = |A_2 \cup B_2| + n_{AB} = n$, it contains a red C_n by Theorem 13. □

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