A Multipartite Ramsey Number for Odd Cycles

Fabrício Siqueira Benevides[1,2]

[1] CAPES FOUNDATION, MINISTRY OF EDUCATION OF BRAZIL
CAIXA POSTAL 250, BRASILIA, DF, 70040.020, BRAZIL

[2] UNIVERSITY OF MEMPHIS
MEMPHIS, TENNESSEE 38152
E-mails: fabriciosb@gmail.com; fbenevds@memphis.edu

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Abstract: In this article we study multipartite Ramsey numbers for odd cycles. Our main result is the proof that a conjecture of Gyárfás et al. (J Graph Theory 61 (2009), 12–21), holds for graphs with a large enough number of vertices. Precisely, there exists n_0 such that if $n \geq n_0$ is a positive odd integer then any two-coloring of the edges of the complete five-partite graph $K_{(n-1)/2,(n-1)/2,(n-1)/2,(n-1)/2,1}$ contains a monochromatic cycle of length n. © 2011 Wiley Periodicals, Inc. J Graph Theory 71: 293–316, 2012

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1. INTRODUCTION

For graphs $L_1, \ldots, L_k$, the Ramsey number $R(L_1, \ldots, L_k)$ is the minimum integer N such that for any edge-coloring of K_N, the complete graph on N vertices, by k colors, there exists a color i for which the corresponding color class contains L_i as a subgraph. Ramsey numbers have been studied by many authors for many classes of graphs. Here, we are interested in aspects of Ramsey numbers for cycles. The particular case where the graphs L_1, L_2 are cycles of length n, denoted by C_n was raised by

Bondy and Erdős [6] and it was fully solved by Faudree and Schelp [8], and independently by Rosta [18]. They have proved, among other things, that

$$R(C_n, C_n) = \begin{cases} 6 & n = 3 \text{ or } 4, \\ 2n-1 & \text{if } n \text{ is odd, } n \geq 5, \\ 3n/2-1 & \text{if } n \text{ is even, } n \geq 6. \end{cases}$$

Bondy and Erdős [6] conjectured that if $n > 3$ is odd, then

$$R(C_n, C_n, C_n) = 4n-3. \tag{1}$$

Kohayakawa et al. (submitted) proved that there exists an n_0 such that Equation (1) holds for every n odd with $n > n_0$.

The case when n is even differs from the case when n is odd. Benevides and Skokan [2] proved that there exists an integer n_1 such that for every even $n > n_1$

$$R(C_n, C_n, C_n) = 2n. \tag{2}$$

Recently, there has been interest to see what happens to the Ramsey numbers when we allow fixed edge deletions from the complete graph K_N. In particular, if we delete complete subgraphs K_r.

For example, a tripartite version of the Gerencsér-Gyárfás Theorem was given by Gyárfás et al. [13], i.e., it was proved that the Ramsey number for a path is about the same when two-colorings of a complete graph or a balanced complete tripartite graph are considered. In an article of Nikiforov and Schelp [17], it was shown, among other things, that for any odd $n \geq 5$ if we delete the edges of a complete subgraph of order $(n-1)/2$ from the complete graph of order $2n-1$ and we two-color the rest, we can still guarantee a monochromatic C_n. And in a recent article of Gyárfás et al. [12], the following theorem in the same direction was proved.

Theorem 1. *For all $0<\eta<1/2$, there exists an $n_1 = n_1(\eta)$ with the following properties. For any odd integer $n > n_1$, in any two-coloring of the edges of the complete five-partite graph of order $(2+\eta)n$ with five parts of size $g(1), g(2), g(3), g(4)$ and $g(5)$, where we have $n/2 \geq g(1) \geq g(2) \geq g(3) \geq g(4) \geq g(5) \geq \eta n$, there is a monochromatic C_n.*

In this article we prove that an exact result generalizing the above theorem holds for sufficiently large n. This result was conjectured in the same article where Theorem 1 appeared [12]. More precisely, we prove the following main theorem.

Theorem 2. *There exists n_2 such that, for any odd integer $n \geq n_2$, in any two-coloring of the edges of the complete five-partite graph $K_{(n-1)/2,(n-1)/2,(n-1)/2,(n-1)/2,1}$ there is a monochromatic C_n.*

Notice that the graph we are coloring above is obtained from a K_{2n-1} by making four big 'holes' of order $(n-1)/2$ each. This is somewhat surprising and sharp, since if we had made only a single hole of order $(n+1)/2$, instead of four holes of order $(n-1)/2$, there is no guarantee that we can find a monochromatic C_n. In fact, let $A \subset V = V(K_{2n-1})$ with $|A| = (n+1)/2$ and consider the graph obtained by the removal of the edges spanned by A from K_{2n-1}. Split the vertices $V \setminus A$ into two sets B and C with $|B| = (n-1)/2$ and $|C| = n-1$. Color all the edges within B, within C, and between A and B by *red*; and color the remaining edges, i.e., those between $A \cup B$ and C, by *green*. It is easy to see that there is no monochromatic C_n.

Furthermore, the total number of edges removed is almost one-fourth of the total number of edges, which seems to be a substantial number of edges to remove without losing the property that we find a monochromatic C_n among the remaining colored edges. In fact, one can think about the removed edges as a third color, but one in which it is not satisfying to find a monochromatic cycle. From Equation (1), if we had no restrictions on that third color, we would need $4n-3$ vertices (instead of $2n-1$) in order to guarantee a monochromatic C_n.

2. NOTATION

Our notation is standard. Nevertheless, we emphasize a few points here.

We note that the subscripts for an absolute or relative constant in a theorem/lemma are equal to the reference number of the theorem/lemma. This makes it much easier for the reader to find the place where a constant is defined.

For graphs, unless otherwise stated, the first subscripts indicate the number of vertices, e.g., C_n is the cycle with n vertices and P_n is the path with n vertices. The *length* of a path is a number of its edges and, if x is its first vertex and x' is its last vertex, then we call it an (x,x')-path. Given a set X of vertices of a graph G, $G[X]$ denotes the subgraph induced by the edges with both ends in X and $G \backslash X$ denotes the subgraph obtained by deleting the vertices of X and the edges incident to the deleted vertices.

For a multipartite graph G, we shall work with *its multipartite complement*, $\overline{G}$, defined as the graph we obtain from the usual complement of G by deleting all edges within the classes in the given vertex partition.

Given two disjoint nonempty sets of vertices X and Y, $E(X,Y)$ denotes the set of all the edges with one end in X and the other one in Y. We also set $e(X,Y) = |E(X,Y)|$ and

$$d(X,Y) = \frac{e(X,Y)}{|X||Y|}.$$

We denote by $G[X,Y]$ the bipartite subgraph of G with bipartition $X \cup Y$ and the edge set $E(X,Y)$, and in general for disjoint sets $X_1, X_2, \ldots, X_k$ we denote by $G[X_1, X_2, \ldots, X_k]$ the multipartite graph induced by the edges of G from X_i to X_j for every $i \neq j$.

Whenever we speak about colorings, we mean edge-colorings. Mostly we use two colors, *red* and *green*. Sometimes a third color will be needed and for that we use *blue*.

The subgraphs induced by the edges of a given color are indicated by superscripts: G^r is the *red* subgraph of G. But for the corresponding graph theoretical parameters such as number of edges or degrees we use subscripts: $e_r(X,Y)$ denotes the number of *red* edges joining X to Y in an edge-colored graph. If an edge xy of G is *red*, we say that y is a *red* neighbor of x (and vice-versa). For a vertex x, $N(x)$ denotes the set of all vertices adjacent to x and we set $\deg(x,Y) = |N(x) \cap Y|$ (the degree of x to Y) and $\deg_r(x,Y) = |N_r(x) \cap Y|$ (the *red* degree of x to Y). The maximum degree of a vertex in G is denoted by $\Delta(G)$ and the minimum degree is denoted by $\delta(G)$.

A graph G_n is called γ-dense if it has at least $\gamma\binom{n}{2}$ edges. A bipartite graph with parts of order k and ℓ is γ-dense if it contains at least $\gamma k \ell$ edges.

A matching M in a graph G is a set of pairwise vertex-disjoint edges. We denote the number of edges in M by $e(M)$. A *connected matching* is a matching M such that all the edges of M are in the same component C of G. We say that M is an *odd connected matching*, if the component C is not bipartite. Finally, we say that M is a *monochromatic connected matching*, if it is a connected matching within the graph induced by one of the colors.

3. EXTREMAL COLORINGS AND STABILITY

In this article, we will use a variant of a stability theorem of Gyárfás, Ruszinkó, Sárközy, and Szemerédi [10, 11], stated by Benevides and Skokan [1, 2]. But before we can state this theorem we need to define particular (extremal) colorings. It is convenient, as we shall notice later, to consider three-multicolorings instead of three-colorings. In a *three-multicoloring* of a graph G, some of its edges can be assigned more than one color. We say that an edge is *exclusive of color c*, for $c \in \{(\text{r})\text{ed}, (\text{g})\text{reen}, (\text{b})\text{lue}\}$, if it is assigned color c only. We denote by G^{c^*} the subgraph induced by the edges exclusively colored by C.

Now, we define the three types of colorings:

Coloring 1 ($EC_1(\alpha,\delta)$ type). *A three-multicoloring of a graph G is of type $EC_1(\alpha,\delta)$, where $0 \le \alpha, \delta < 1$, if there exists a partition $A \cup B \cup C \cup D$ of $V(G)$ such that*

(*a*) $|A|, |B|, |C|, |D| \ge (1-\alpha)|V(G)|/4$.
(*b*) *The bipartite graphs $G^{r^*}[A,B], G^{r^*}[C,D], G^{g^*}[A,D], G^{g^*}[B,C], G^{b^*}[A,C]$, and $G^{b^*}[B,D]$ are $(1-\delta)$-dense.*

Coloring 2 ($EC_2(\alpha,\delta)$ type). *A three-multicoloring of a graph G is of type $EC_2(\alpha,\delta)$, where $0 \le \alpha, \delta < 1$, if there exists a partition $A \cup B \cup C \cup D$ of $V(G)$ such that*

(*a*) $|A|, |B|, |C|, |D| \ge (1-\alpha)|V(G)|/4$.
(*b*) *The bipartite graphs $G^{r^*}[A,B]$, $G^{g^*}[A \cup B, C]$, and $G^{b^*}[A \cup B, D]$ are $(1-\delta)$-dense.*

Coloring 3 ($EC_3(\mu,c_1,c_2,\delta)$ type). *A three-multicoloring of a graph G is of type $EC_3(\mu,c_1,c_2,\delta)$, where $0 \le \mu,\ c_1,\ c_2,\ \delta < 1$, if there exists a partition $A \cup B \cup C \cup D$ of $V(G)$ such that*

(*a*) $|A|, |B|, |C| \ge (1-c_1\mu)|V(G)|/4,\ |D| \ge \mu|V(G)|/4$.
(*b*) $|A| \ge \max\{|B|, |C|, |D|\} + \mu|V(G)|/4,\ |A \cup D| \le (1+c_2\mu)|V(G)|/2$.
(*c*) *The bipartite graphs $G^{r^*}[A,B]$, $G^{r^*}[C,D]$, $G^{r^*}[A,D]$, $G^{g^*}[B,C]$, $G^{b^*}[A,C]$ and $G^{b^*}[B,D]$ are $(1-\delta)$-dense (Fig. 1).*

Now we can state the variant of the stability lemma of Gyárfás et al. [10, 11].

Theorem 3 (Benevides [1] and Benevides and Skokan [2]). *Given $\alpha_0 > 0$ and $\mu_0 > 0$, there exists positive reals η_3, β_3, and μ_3, $\mu_3 < \mu_0$, such that for all $\beta < \beta_3$ there exists a positive integer $n_3 = n_3(\beta, \mu_3, \eta_3, \alpha_0)$ such that the following holds. If $n \ge n_3$ and a $(1-\beta)$-dense graph G_n of order n is three-multicolored, then one of the following cases*

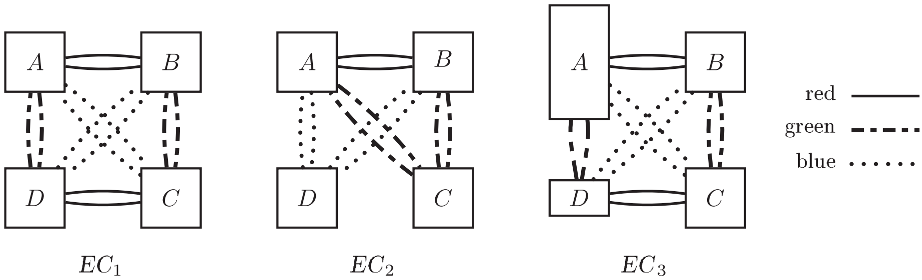

FIGURE 1. Three different types of colorings.

occurs:

(*a*) G_n *contains a monochromatic connected matching of with least* $(1/4+\eta_3)n$ *edges*;
(*b*) *the coloring is of type* $EC_1(\alpha_0,\alpha_0)$, *or* $EC_2(\alpha_0,\alpha_0)$, *or* $EC_3(\mu_3,0.7,0.2,\beta^{1/3})$.

Remark 4. In a multicoloring, we consider a set E of edges monochromatic if there is a color c such that all edges in E have been colored with c. However, note that we do not require the edges in E to be colored exclusively by c.

The proof of Theorem 3 is essentially the same as in [10] and it can be found in [1]. This theorem was used in [10] to compute $R(P_n,P_n,P_n)$ and in [2] to compute $R(C_n,C_n,C_n)$ when n is even. It basically says that either we find a large monochromatic connected matching or the coloring of the graph can be well described. Later in this article, we will use this theorem to prove Theorem 13 which, in turn, will be used in the proof of Theorem 2. Theorem 13 involves other two different types of colorings, this time, two-multicolorings of a four-partite graph. We define those colorings here, but we will state Theorem 13 only when needed, in Section 5.

Coloring 4 ($EC_A(\alpha,\delta)$ type)**.** *A two-multicoloring of a four-partite graph G is of type* $EC_A(\alpha,\delta)$*, where* $0\le\alpha,\delta<1$*, if there exists disjoint sets of vertices A, B, C and D such that*

(*a*) $|A|,|B|,|C|,|D|\ge(1-\alpha)|V(G)|/4$ *and each of A, B, C and D is an independent set.*
(*b*) *The bipartite graphs* $\overline{G^{g^*}}[A,D]$ *and* $\overline{G^{g^*}}[B,C]$ *have maximum degree at most* $\delta|V(G)|$.
(*c*) *The bipartite graphs* $\overline{G^{r^*}}[A,B]$ *and* $\overline{G^{r^*}}[C,D]$ *have maximum degree at most* $\delta|V(G)|$.

Remark 5. Condition (a) implies that at most $\alpha|V(G)|$ vertices do not belong to $A\cup B\cup C\cup D$.

Coloring 5 ($EC_B(\alpha,\delta)$ type)**.** *A two-multicoloring of a four-partite graph G, whose vertex partition* (*into independent sets*) *is given, say* $V(G)=U_1\cup U_2\cup U_3\cup U_4$, *is of type* $EC_B(\alpha,\delta)$, *where* $0\le\alpha,\delta<1$, *if there exists disjoint sets X,* $Y\subseteq V(G)$ *for which,*

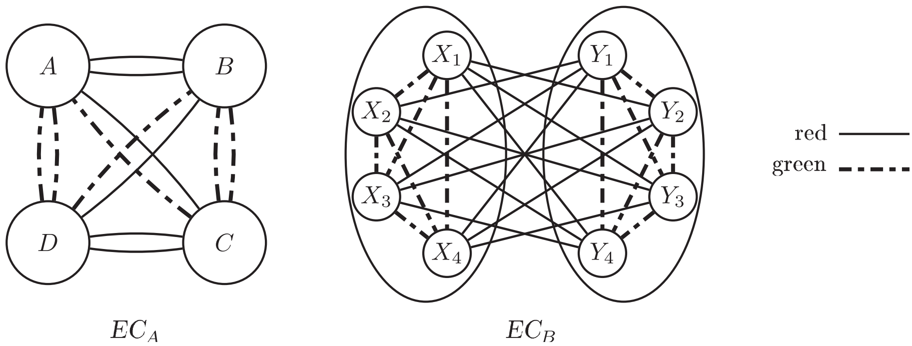

FIGURE 2. Two other types of colorings.

letting $X_i = U_i \cap X,\ Y_i = U_i \cap Y$ $(1 \le i \le 4)$, *we have*

(*a*) $|X|,|Y| \ge (1-\alpha)|V(G)|/2$.

(*b*) *For* $1 \le i \le 4$, *the bipartite graph* $\overline{G^{r^*}}[X_i, \bigcup_{j\ne i} Y_j]$ *has maximum degree at most* $\delta|V(G)|$.

(*c*) *For* $1 \le i \le 4$, *the bipartite graph* $\overline{G^{r^*}}[Y_i, \bigcup_{j\ne i} X_j]$ *has maximum degree at most* $\delta|V(G)|$.

(*d*) *The multipartite graphs* $\overline{G^{g^*}}[X_1,X_2,X_3,X_4]$ *and* $\overline{G^{g^*}}[Y_1,Y_2,Y_3,Y_4]$ *have maximum degree at most* $\delta|V(G)|$.

Remark 6. Condition (a) implies that at most $\alpha|V(G)|$ vertices do not belong to $X \cup Y$ (Fig. 2).

The remainder of this article is organized as follows: in Section 4 we present Szemerédi's Regularity Lemma; in Section 5 we state (without proofs) our main tools, one theorem and two lemmas, and use them to prove Theorem 2; in Sections 6 and 7, we give the missing proofs. Finally, in Section 8 we post some concluding remarks and conjectures.

4. REGULARITY LEMMA FOR GRAPHS

Szemerédi's Regularity Lemma [19] asserts that each graph of positive edge-density can be approximated by the union of a bounded number of random-like bipartite graphs. Before it can be stated formally, the concept of *ε-regular pairs* needs to be defined.

Definition 7. Let $G=(V,E)$ be a graph and let $0<\varepsilon\le 1$. We say that a pair (A,B) of two disjoint subsets of V is ε-regular (with respect to G) if

$$|d(A',B')-d(A,B)|<\varepsilon$$

holds for any two subsets $A' \subset A$, $B' \subset B$ with $|A'|>\varepsilon|A|$, $|B'|>\varepsilon|B|$.

This definition states that a regular pair has uniformly distributed edges. In the next section, we will make implicitly use of the following fact about regular pairs.

Fact 8. *Let G be a bipartite graph with bipartition $V(G)=V_1\cup V_2$ such that the pair (V_1,V_2) is ε-regular with density $d=d(V_1,V_2)$. Then all but at most $\varepsilon|V_1|$ vertices $v\in V_1$ satisfy* $\deg(v)\geq(d-\varepsilon)|V_2|$.

The next lemma about regular pairs is a slightly stronger version of Claim 3 from [16]. The version in [16] is the case where $\gamma=1$. Both statements have the same proof that we omit here.

Lemma 9. *For every $0<\gamma<1$, there exists an n_9 such that for every $n>n_9$ the following holds: Let G be a bipartite graph with bipartition $V(G)=V_1\cup V_2$ such that $|V_1|,|V_2|\geq n$. Furthermore, let the pair (V_1,V_2) be ε-regular with density at least $\gamma/4$ for some ε satisfying $0<\varepsilon<\gamma/100$. Then for every ℓ, $1\leq\ell\leq n-5\varepsilon n/\gamma$, and for every pair of vertices $v'\in V_1$, $v''\in V_2$ satisfying* $\deg(v')$, $\deg(v'')\geq\gamma n/5$, *G contains a path of length $2\ell+1$ connecting v' and v''.*

The regularity lemma of Szemerédi [19] enables us to partition the vertex set $V(G)$ of a graph G into $t+1$ sets $V_0\cup V_1\cup\cdots\cup V_t$ in such a way that almost all the pairs (V_i,V_j) satisfy Definition 7 for some small ε. Its precise statement, extended to more than one graph, is as follows.

Theorem 10. *For every $\varepsilon>0$ and for all positive integers s and m, there exist integers $N_{10}=N_{10}(\varepsilon,s,m)$ and $M_{10}=M_{10}(\varepsilon,s,m)$ with the following property: for all graphs $G_1,\ldots,G_s$ with the same vertex set V, $|V|\geq N_{10}$, there is a partition of V into $t+1$ sets*

$$V=V_0\cup V_1\cup\cdots\cup V_t$$

such that

(*a*) $m\leq t\leq M_{10}$,
(*b*) $|V_0|\leq\varepsilon n$, $|V_1|=\cdots=|V_t|$, *and*
(*c*) *all but at most $\varepsilon\binom{t}{2}$ pairs (V_i,V_j), $1\leq i<j\leq t$, are ε-regular with respect to each G_k, $1\leq k\leq s$.*

Remark 11. The original regularity lemma refers to the case $s=1$. The proof is (basically) the same for an arbitrary but fixed number s of graphs. This version is used, for example, in [7], and formulated in the survey [14].

Remark 12. The sets V_i in the partition given by this lemma are called clusters. When the lemma is applied to a multipartite graph, we can assume that each of those clusters is contained in one of the parts.

5. MAIN TOOLS AND PROOF OF THEOREM 2

Our main tool is the following theorem, whose proof we postpone to Section 6.

Theorem 13. *Given α_1, there exists positive reals η_{13}, $\beta_{13}>0$ such that for every $\beta<\beta_{13}$ there exists $n_{13}=n_{13}(\beta,\eta_{13})$ such that for any $n>n_{13}$ the following holds: if G is a four-partite graph on n vertices such that each part has at least $(1/4-\beta)n$ vertices and its multipartite complement $\overline{G}$ satisfies $\Delta(\overline{G})\leq\beta n$, then for any two-multicoloring of G,* *either we find an odd connected monochromatic matching of size at least* $(1/4+\eta_{13})n$ *edges or the coloring is of type* $EC_A(\alpha_1,\alpha_1)$ *or* $EC_B(\alpha_1,\alpha_1)$.

We will also need the following two lemmas, which we will prove in later sections as well.

Lemma 14. *For* $n\geq 3$ *odd, let* $G=K_{(n-1)/2,(n-1)/2,(n-1)/2,(n-1)/2,1}$, *let* u *be its only vertex of degree* $2n-2$ *and let* $H=G\setminus\{u\}$. *There exists* $\alpha_{14}>0$ *such that, for all* $\alpha\leq\alpha_{14}$ *and* $\delta\leq\alpha$, *there is a positive integer* n_{14} *with the following property: for every odd* $n\geq n_{14}$, *every two-coloring of* G, *such that the induced coloring in* H *is of type* $EC_A(\alpha,\delta)$, *contains a monochromatic* C_n.

Lemma 15. *For* $n\geq 3$ *odd, let* $G=K_{(n-1)/2,(n-1)/2,(n-1)/2,(n-1)/2,1}$, *let* u *be its only vertex of degree* $2n-2$ *and let* $H=G\setminus\{u\}$. *There exists* $\alpha_{15}>0$ *such that, for all* $\alpha\leq\alpha_{15}$ *and* $\delta\leq\alpha$, *there is a positive integer* n_{15} *with the following property: for every odd* $n\geq n_{15}$, *every two-coloring of* G, *such that the induced coloring in* H *is of type* $EC_B(\alpha,\delta)$, *contains a monochromatic* C_n.

Now we restate Theorem 2 for easy reference and we give a sketch of the proof before the full proof is shown.

Theorem 2. *There exists* n_2 *such that, for any odd integer* $n\geq n_2$, *in any two-coloring of the edges of the complete five-partite graph* $K_{(n-1)/2,(n-1)/2,(n-1)/2,(n-1)/2,1}$ *there is a monochromatic* C_n.

We shall consider a two-coloring of the graph $G=K_{(n-1)/2,(n-1)/2,(n-1)/2,(n-1)/2,1}$, say (G^r,G^g), where n is odd and $n>n_0$. Let u be the (only) vertex of G of degree $2n-2$. We apply the regularity lemma (Theorem 10) with carefully chosen ε, m and with $s=2$ to the graphs $G^r\setminus\{u\},G^g\setminus\{u\}$ and obtain a partition $V_0\cup V_1\cup\cdots\cup V_t$ of $V(G)\setminus\{u\}$ satisfying conditions (a)–(c) in Theorem 10. Using this partition, we define the so-called reduced graph R and also an appropriate two-multicoloring of its edges: the vertex set of R is $\{1,\ldots,t\}$, we have an edge between i and j if and only if (V_i,V_j) has density at least $\varepsilon^{1/3}/2$ and is an ε-regular pair with respect to G^r and G^g, and an edge ij is colored by *red* (resp. *green*) if $G^r[V_i,V_j]$ (resp. $G^g[V_i,V_j]$) has edge density at least $\varepsilon^{1/3}/4$.

By Remark 12, we can assume that the reduced graph R is four-partite. Then, we apply Theorem 13 to R, which will lead us to one of three cases: either R has a monochromatic connected odd matching of a certain size or its two-multicoloring is of type EC_A or of type EC_B. In the first case, we use the matching in R to find a copy of C_n in G of the same color of the matching by applying Lemma 9 many times. This will be very similar to what is done in Section 5 of [2], except that here we will need to find an odd cycle while in [2] an even one was needed. In the other two cases, we prove that the original coloring of G must be of the same type as the coloring of R. In this case, we apply Lemma 14 or Lemma 15 to the original graph G to find a monochromatic C_n in G.

Proof. We start by choosing some parameters.

Let $\alpha_1=\min\{(\alpha_{14}/10)^2,(\alpha_{15}/10)^2,1/20\}$ so that, in particular, we can input $\delta=\alpha=10\sqrt{\alpha_1}$ to Lemmas 14 and 15 and get $n_{14}=n_{14}(10\sqrt{\alpha_1},10\sqrt{\alpha_1})$ and $n_{15}=n_{15}(10\sqrt{\alpha_1},10\sqrt{\alpha_1})$. Passing α_1 to Theorem 13, we obtain $\eta_{13}=\eta_{13}(\alpha_1)$ and $\beta_{13}=\beta_{13}(\alpha_1)$.

We define

$$\varepsilon = \frac{1}{2}\min\left\{(\beta_{13}/2)^2, 1/10^6, \frac{\alpha_1^3}{1000}, \frac{\eta_{13}^2}{2000}\right\}. \tag{3}$$

Let $\beta = 2\sqrt{\varepsilon}$ and notice that $\beta < \beta_{13}$. With this β, Theorem 13 yields $n_{13} = n_{13}(\beta, \eta_{13})$. We also set $m = \max\{2n_{13}, 1/\varepsilon\}$ and, from Theorem 10, we obtain $N_{10} = N_{10}(\varepsilon, 2, m)$ and $M_{10} = M_{10}(\varepsilon, 2, m)$. Then we may finally choose

$$n_2 = \max\left\{n_{14}, n_{15}, N_{10}, 2M_{10}n_9, \frac{M_{10}^2}{\varepsilon^{1/3}}\right\}. \tag{4}$$

Consider any two-coloring (G^r, G^g) of $G = K_{(n-1)/2,(n-1)/2,(n-1)/2,(n-1)/2,1}$ with n odd and $n > n_2$. We denote $V(G) = U_1 \cup U_2 \cup U_3 \cup U_4 \cup \{u\}$, where U_1, U_2, U_3, U_4 are the independent sets of order $(n-1)/2$ and u is the (only) vertex of degree $2n-2$. We apply the regularity lemma (Theorem 10) to the pair of graphs $G^r \setminus \{u\}$ and $G^g \setminus \{u\}$, with parameters ε and m chosen as above (and $s = 2$).

Let $V = V(G) = V_0 \cup V_1 \cup \cdots \cup V_t$ be the partition guaranteed by this lemma, thus satisfying

(a) $m \le t \le M_{10}$,
(b) $|V_0| \le \varepsilon(2n-2)$, $|V_1| = \cdots = |V_t|$, and
(c) all but at most $\varepsilon\binom{t}{2}$ pairs (V_i, V_j), $1 \le i < j \le t$, are ε-regular with respect to both G^r and G^g.

By Remark 12, we can assume that each V_k, $1 \le k \le t$, is a subset of U_i for some i, $1 \le i \le 4$.

Now we define a *reduced graph* R in the following way: the vertex set of R is $\{1, \ldots, t\}$ and we have an edge between vertices i and j if and only if V_i and V_j are contained in different sets of the partition $\{U_1, U_2, U_3, U_4\}$ and (V_i, V_j) is an ε-regular pair with respect to both G^r and G^g. Note that, by definition, R is a four-partite graph, say $V(R) = X_1 \cup X_2 \cup X_3 \cup X_4$, with $X_i = \{k : V_k \subset U_i, 1 \le k \le t\}$. It is easy to see that, all sets X_i have approximately the same order. More precisely, if we denote $t_i = |X_i|$, then $t_i \ge (1/4 - \varepsilon)t$, for $1 \le i \le 4$. In fact, for any $1 \le i \le 4$ and for an arbitrary $k \ne 0$, the above property (b) implies that

$$t_i \frac{2n-2}{t} \ge t_i |V_k| = |U_i| - |U_i \cap V_0| \ge \left(\frac{1}{4} - \varepsilon\right)(2n-2)$$

and the previous statement follows.

It is convenient here to work on graphs with high degree (rather than simply on dense graphs). So, we start by cleaning up R. We throw away a (small) set of vertices that do not have high degree. Let $F = \{v \in V(\overline{R}) : \deg_{\overline{R}}(v) \ge \sqrt{\varepsilon} t\}$, where $\overline{R}$ is the multipartite complement of R. We have

$$|F|\sqrt{\varepsilon} t \le 2e(\overline{R}) \le 2\varepsilon\binom{t}{2},$$

where the second inequality follows from property (c) above. Then, $|F| \le \sqrt{\varepsilon}(t-1) < \sqrt{\varepsilon} t$. We consider the graph H induced by $V(R) \setminus F$ and denote $t' = |V(H)|$ and $X_i' = X_i \setminus F$.

Clearly, $t' \geq (1-\sqrt{\varepsilon})t$. Therefore

$$\Delta(\overline{H}) \leq \sqrt{\varepsilon}t \leq \frac{\sqrt{\varepsilon}}{1-\sqrt{\varepsilon}}t' \leq 2\sqrt{\varepsilon}t' = \beta t'$$

and

$$|X_i'| \geq (1/4-\varepsilon)t - \sqrt{\varepsilon}t \geq (1/4-2\sqrt{\varepsilon})t \geq (1/4-2\sqrt{\varepsilon})t' = (1/4-\beta)t'.$$

We define a two-multicoloring (H^r, H^g) of H in the following way: for $c \in \{r, g\}$, and $ij \in E(H)$ we put ij into H^c if $e_c(V_i, V_j) \geq \varepsilon^{1/3}|V_i||V_j|/4$. Because $t' \geq (1-\sqrt{\varepsilon})t \geq (1-\sqrt{\varepsilon})m \geq m/2 \geq n_{13}$, by the above conditions on $|X_i'|$ and $\Delta(\overline{H})$ and since $\beta < \beta_{13}$, we can apply Theorem 13 (with parameters α_1, η_{13}, β) to H so that either we find an odd monochromatic connected matching M of size $t_1 \geq (1/4+\eta_{13})t'$ or we conclude that the coloring of H is of type $EC_A(\alpha_1, \alpha_1)$ or $EC_B(\alpha_1, \alpha_1)$. We analyze each of these three cases below.

Case 1. There is an odd monochromatic connected matching M of size t_1 in H, $t_1 \geq (1/4+\eta_{13})t'$. Note that

$$(1/4+\eta_{13})t' \geq (1/4+\eta_{13})(1-\sqrt{\varepsilon})t \geq (1/4+\eta_{13}/2)t.$$

Without loss of generality assume that M is *red* and let a_ib_i, $0 \leq i < t_1$, be all the edges of M. Let K be the *red* (non-bipartite) component containing M.

First, we will find, in K, a closed *red* walk of odd length which contains all edges of M. Take a spanning tree T of K such that $E(T)$ contains all edges of M (this can be done, for example, using Krushal's algorithm, i.e., starting with the edges of K and carefully adding new edges until we get a spanning tree). Let Z be the closed minimal walk containing all the edges of T. Such a walk contains each edge of T exactly twice; therefore, it has some even length which is smaller than $2t$.

Consider some arbitrary vertex r of T and look at the levels of T as a rooted tree with root r. Since K is a non-bipartite component, there must exists a *red* edge $xy \notin E(T)$, such that x and y are in levels of same parity, i.e., the lengths of the unique paths from x to r and from y to r in T have the same parity. Therefore, the unique path P_{xy} from x to y contained in Z has even length. So, we can construct a walk W by taking Z and replacing P_{xy} by the edge xy. It is clear that W is a closed walk, it has odd length and it contains every edge of M (at least once), as desired. Let $W = i_1 i_2 \ldots i_\ell i_1$.

To finish this case, one uses the walk W and standard regularity arguments to build a *red* C_n in the original graph G. The same technique was used earlier in results about even cycles by Benevides and Skokan [2]. Similar ideas are also applied by Gyárfás et al. [12], by Gyárfás and Szemerédi [10] and even earlier by Łuczak [16]. Nevertheless, we do the all the computations here for completeness.

For each j, with $0 \leq j \leq \ell$, we say that a vertex in the set V_{i_j} is "good" if it has at least $\varepsilon^{1/3}|V_{i_{j-1}}|/4$ *red* neighbors in each of $V_{i_{j-1}}$ and $V_{i_{j+1}}$, where we set $V_{i_0} = V_{i_\ell}$ and $V_{i_{\ell+1}} = V_{i_1}$; and we say that a vertex is "bad" otherwise. Note that for any j, by Fact 8 applied to $(V_{i_j}, V_{i_{j+1}})$ and to $(V_{i_j}, V_{i_{j-1}})$, at most $2\varepsilon|V_{i_j}|$ vertices of V_{i_j} are bad. The next step in the proof is to construct a (small) cycle $\tilde{C} = v_{i_1} v_{i_2} \ldots v_{i_\ell}$ with $v_{i_j} \in V_{i_j}$ such that all its vertices are good. We emphasize that while we may have $V_{i_k} = V_{i_j}$, for some numbers k, j with $k \neq j$, the vertices v_{i_j} of C are chosen to be pairwise distinct. Starting with any good vertex v_{i_1} in V_{i_1}, one can greedily build a path $P = v_{i_1} v_{i_2} \dots v_{i_{\ell-2}}$ with $v_{i_j} \in V_{i_j}$ such that all its vertices are good: for each $1 \le j \le \ell - 3$ let $v_{i_{j+1}}$ be any *red* good neighbor of v_{i_j} in $V_{i_{j_1}}$. Such good neighbor exists as

$$\frac{\varepsilon^{1/3}}{4} |V_{i_{j-1}}| - 2\varepsilon |V_{i_j}| - j \ge \varepsilon |V_{i_j}| > 1.$$

To prove the existence of the cycle $\tilde{C}$ from the existence of P, consider the set A of good neighbors of $v_{i_{\ell-2}}$ in $V_{i_{\ell-1}}$ and the set B of good neighbors of v_{i_1} in V_{i_ℓ}. As before, we have $|A|, |B| \ge \varepsilon |V_{i_1}|$. Because the pair $(V_{i_{s-1}}, V_{i_s})$ is ε-regular with density at least $\varepsilon^{1/3}/4$, it follows that $G[A,B]$ has density at least $\varepsilon^{1/3}/4 - \varepsilon \ge \varepsilon^{1/3}/5$. Therefore, $G[A,B]$ has at least $\varepsilon^{1/3}|A||B|/5$ edges. This number is greater than one by the choice of n. Letting $v_{i_{s-1}} v_{i_s}$ be any edge in $G[A,B]$, we have that $v_{i_1} v_{i_2} \dots v_{i_{s-1}} v_{i_s}$ is a cycle as desired.

Finally, we show that we can use Lemma 9, to replace the edges of $\tilde{C}$ corresponding to edges of M by long paths in such a way that the resulting larger cycle is a C_n. For each edge $a_k b_k$ of M, we choose a natural number ℓ_k satisfying

$$1 \le \ell_k \le (1 - 10\varepsilon^{2/3}) \min\{|V_{a_k}| - 2t, |V_{b_k}| - 2t\}$$

in such a way that

$$\sum_{k=0}^{t_1 - 1} 2\ell_k = n - \ell.$$

This is possible because $n - \ell$ is even, $n - \ell \ge 2t \ge 2t_1$ and $\sum_{k=0}^{t_1-1} 2\ell_k$ can attain any even value between $2t_1$ and

$$\begin{aligned}
&\sum_{i=0}^{t_1-1} 2(1 - 10\varepsilon^{2/3}) \min\{|V_{a_k}| - 2t, |V_{b_k}| - 2t\} \\
&\quad \ge 2t_1 (1 - 10\varepsilon^{2/3}) \left(\frac{(1-\varepsilon)(2n-2)}{t} - 2t \right) \\
&\quad \ge \left(\frac{1}{2} + \eta_{13} \right) t (1 - 10\varepsilon^{2/3}) \frac{(1-2\varepsilon)(2n-2)}{t} \\
&\quad \ge (1 + \eta_{13}) n > n - \ell.
\end{aligned}$$

Finally, we set $V'_{a_k} = (V_{a_k} \setminus \tilde{C}) \cup \{v_{a_k}\}$, $V'_{b_k} = (V_{b_k} \setminus \tilde{C}) \cup \{v_{b_k}\}$ and notice that

$$|V'_{a_k}| \ge |V_{a_k}| - |\tilde{C}| \ge |V_{a_k}| - 2t \ge |V_{a_k}| - 2M_{10} \ge \frac{|V_{a_k}|}{2} \ge \frac{(1-\varepsilon)(2n-2)}{2M_{10}} > n_9$$

and, similarly

$$|V'_{b_k}| \ge \frac{|V_{b_k}|}{2} > n_9.$$

Hence, $G^r[V'_{a_k}, V'_{b_k}]$ is (2ε)-regular with density at least $\varepsilon^{1/3}/4 - \varepsilon > \varepsilon^{1/3}/5$ and we can apply Lemma 9 (with $\gamma = \varepsilon^{1/3}$) to $G^r[V'_{a_k}, V'_{b_k}]$. Since

$$1 \le \ell_k \le (1-5\varepsilon^{2/3})\min\{|V_{a_k}|-2t, |V_{b_k}|-2t\} \le \left(1-5\frac{2\varepsilon}{\varepsilon^{1/3}}\right)\min\{|V'_{a_k}|, |V'_{b_k}|\}$$

there exists a path P_{a_k,b_k} of length $2\ell_k+1$ that starts at v_{a_k}, ends at v_{b_k}, and consists only of edges in $G^r[V'_{a_k}, V'_{b_k}]$. In $\tilde{C}$, we replace each edge $v_{a_k}v_{b_k}$ by the path P_{a_k,b_k}. This yields a *red* cycle of length $\ell - t_1 + \sum_{k=0}^{t_1-1}(2\ell_k+1) = n$.

Case 2. (H^r, H^g) is a coloring of type $EC_A(\alpha_1, \alpha_1)$. We will show that this implies that $(G^r \setminus \{u\}, G^g \setminus \{u\})$ is of type $EC_A(10\sqrt{\alpha_1}, 10\sqrt{\alpha_1})$. Let A, B, C, D be subsets of $V(H)$ satisfying conditions (a)–(c) of $EC_A(\alpha_1, \alpha_1)$. It is natural to consider the collection $\{f(A), f(B), f(C), f(D)\}$ of subsets of $V(G)$ given by $f(S) = \bigcup_{j \in S} V_j$ for $S \in \{A, B, C, D\}$. Note that

$$|f(A)| \ge |A|\frac{(1-\varepsilon)(2n-2)}{t} \ge (1-\alpha_1)\frac{t'}{4}\frac{(1-\varepsilon)(2n-2)}{t} \ge (1-2\alpha_1)\frac{2n-2}{4}.$$

Similarly, we obtain that $|f(B)|, |f(C)|, |f(D)| \ge (1-2\alpha_1)(2n-2)/4$. Therefore, condition (a) of $EC_A(10\sqrt{\alpha_1}, 10\sqrt{\alpha_1})$ is satisfied with room to spare. Unfortunately, $\{f(A), f(B), f(C), f(D)\}$ might not satisfy conditions (b) and (c) of $EC_A(10\sqrt{\alpha_1}, 10\sqrt{\alpha_1})$. But we shall prove that we can remove a few (bad) vertices from each $f(S)$, $S \in \{A, B, C, D\}$, so that the resulting sets continue to satisfy (a) and also satisfy (b) and (c).

So, we count how many vertices do not have low degree in one of the bipartite graphs $\overline{G^{g^*}}[f(A), f(D)]$, $\overline{G^{g^*}}[f(B), f(C)]$, $\overline{G^{r^*}}[f(A), f(B)]$ or $\overline{G^{r^*}}[f(C), f(D)]$: we call a vertex bad if its induced degree in any of those graphs is larger than $2\sqrt{\alpha_1}|V(G) \setminus \{u\}| = 2\sqrt{\alpha_1}(2n-2)$. We claim that at most $2\sqrt{\alpha_1}(2n-2)$ vertices are bad.

Fix a vertex $i \in V(H)$ and assume without loss of generality that $i \in A$. We bound the number of *red* edges from V_i to $f(D)$ as follows. Recalling that $f(D) = \bigcup_{j \in D} V_j$, it is enough to bound $e_r(V_i, V_j)$ for each $j \in D$. When $ij \notin H^{g^*}$, we use the trivial bound $|V_i||V_j|$ for $e_r(V_i, V_j)$, but notice that condition (b) of $EC_1(\alpha_1, \alpha_1)$ implies that there are at most $\alpha_1 t'$ such $j's$. While for $ij \in H^{g^*}$ we can conclude that $ij \notin H^r$, thus, from the definition of H^r, $e_r(V_i, V_j) \le \varepsilon^{1/3}|V_i||V_j|/4$. Hence

$$\begin{aligned}
e_r(V_i, f(D)) &\le \sum_{\substack{j \in D \\ ij \notin H^{g^*}}} |V_i||V_j| + \sum_{\substack{j \in D \\ ij \in H^{g^*}}} \frac{\varepsilon^{1/3}}{4}|V_i||V_j| \\
&\le \alpha_1 t'|V_i||V_i| + |D|\frac{\varepsilon^{1/3}}{4}|V_i||V_i| \\
&\le \alpha_1 t|V_i||V_i| + \frac{\varepsilon^{1/3}}{4}t|V_i||V_i| \\
&\le 2\alpha_1|V_i|(2n-2),
\end{aligned}$$

where in the last equation we have used that $|V_i| = |V_j|$ for any $i, j \ge 1$, $t'|V_j| \le 2n-2$ and $\varepsilon^{1/3}/4 \le \alpha_1$.

Therefore, at most $\sqrt{\alpha_1}|V_i|$ vertices of V_i can have more than $2\sqrt{\alpha_1}(2n-2)$ *red* neighbors in $f(D)$. Similarly, at most $\sqrt{\alpha_1}|V_i|$ vertices of V_i can have more than $2\sqrt{\alpha_1}(2n-2)$ *green* neighbors in $f(B)$. Hence, at most $2\sqrt{\alpha_1}|V_i|$ vertices of V_i are bad. Now, if we vary i over all vertices of $V(H)$, we conclude that at most $2\sqrt{\alpha_1}|f(A)\cup f(B)\cup f(C)\cup f(D)|\le 2\sqrt{\alpha_1}(2n-2)$ are bad.

Finally, we define $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, $\tilde{D}$ as the sets obtained from $f(A)$, $f(B)$, $f(C)$, $f(D)$ by removing the bad vertices. We have that

$$|\tilde{A}|\ge|f(A)|-2\sqrt{\alpha_1}(2n-2)\ge(1-10\sqrt{\alpha_1})(2n-2)/4.$$

The same holds for $|\tilde{B}|$, $|\tilde{C}|$, and $|\tilde{D}|$, that is, condition (a) of $EC_A(10\sqrt{\alpha_1},10\sqrt{\alpha_1})$ is satisfied. Clearly, conditions (b) and (c) are satisfied by $\{\tilde{A},\tilde{B},\tilde{C},\tilde{D}\}$ as well. So, the original two-coloring of $G\backslash\{u\}$ is of type $EC_A(10\sqrt{\alpha_1},10\sqrt{\alpha_1})$.

Since $10\sqrt{\alpha_1}<\alpha_{14}$ and $n>n_{14}(10\sqrt{\alpha_1},10\sqrt{\alpha_1})$, we can use Lemma 14 to conclude that there is a monochromatic C_n in G.

Case 3. (H^r,H^g) is a coloring of type $EC_B(\alpha_1,\alpha_1)$.

Again, we can show that this implies that $(G^r\backslash\{u\},G^g\backslash\{u\})$ is of type $EC_B(10\sqrt{\alpha_1},10\sqrt{\alpha_1})$. The idea is exactly the same as in the previous case, hence we omit the technical details here. And similarly as before, since $10\sqrt{\alpha_1}<\alpha_{15}$ and $n>n_{15}(10\sqrt{\alpha_1},10\sqrt{\alpha_1})$, we can use Lemma 15 to conclude that there is a monochromatic C_n in G. ■

6. PROOF OF THEOREM 13

We will need the following two easy lemmas which are variants of lemmas found in [12]. The first lemma is rather trivial but convenient.

Lemma 16. *Assume that $m<n$ is a positive integer, $\Delta(\overline{G_n})<m$ and $H=G_n[A,B]$ is a bipartite subgraph of G_n with $2m<|A|\le|B|$. Then H is a connected subgraph of G_n and contains a matching of size at least $|A|-m$.*

Proof. Two vertices in A (resp. B) have a common neighbor in B (resp. A). Also if $a\in A$, $b\in B$ then any neighbor of a and b have a common neighbor in A. Thus H is a connected subgraph. Moreover any maximum matching M misses fewer than m vertices of A. ■

Lemma 17. *Assume that G is an r-partite graph with N vertices such that $r\ge 2$, and $\Delta(\overline{G})<m$. Suppose that the largest class in the partition of $V(G)$ has at most as many vertices as the sum of the orders of the others. Then G has a matching covering all but at most rm vertices.*

Proof. We do induction on the order of the graph G. If $|G|\le rm$, there is nothing to do, since an empty matching suffices. Let $V(G)=V_1\cup\ldots V_r$, where $|G|>rm$, and assume that $|V_1|\le\cdots\le|V_r|$ where $|V_r|\le|V_1\cup\cdots\cup V_{r-1}|$. Clearly, $|V_r|>m$ and therefore $|V_1\cup\cdots\cup V_{r-1}|>m$. In particular $V_{r-1}\neq\emptyset$. Then we can find an edge xy from V_{r-1} to V_r.

The hypothesis that the largest partite class is at most as large as the sum of the others still holds on the graph $G'=G\backslash\{u,v\}$, though the relative order for the size of the sets

$V_i' = V_i \setminus \{u,v\}$ might change. Now, G' is r'-partite, with $r' \le r$ and, by induction, we can find a matching M' that covers all but $r'm \le rm$ vertices of G'. Finally, $M = M' \cup \{xy\}$ is the matching we are looking for. ■

Remark 18. With just a little more care, one can prove that there is a matching that covers all but at most $2m$ vertices of G. But for this article, we will only use the lemma with $r=4$ and omit unnecessary details.

Corollary 19. *Let G be an r-partite graph with N vertices, $r \ge 2$. Assume that $V(G) = V_1 \cup \cdots \cup V_r$ and V_r is its largest class. Set $k = \max\{|V_r| - \sum_{i=1}^{r-1} |V_i|, 0\}$. Then, if $\Delta(\overline{G}) < m$, then we can find a matching covering all but at most $k + rm$ vertices.*

Proof. Simply remove any k vertices from V_r and use the previous lemma in the resulting graph. ■

Now we can prove Theorem 13.

Proof of Theorem 13. Let $\alpha_1 > 0$ be given. We define two extra parameters by $\alpha_0 = \mu_0 = 1/20$ that will eventually be used as input to Theorem 3 which, in turn, outputs $\eta_3 = \eta_3(\alpha_0, \mu_0)$, $\beta_3 = \beta_3(\alpha_0, \mu_0)$, and $\mu_3 = \mu_3(\alpha_0, \mu_0) < \mu_0 = 1/20$. We also define

$$\eta_{13} = \min\{\eta_3/5, \alpha_1/10\}$$

and

$$\beta_{13} = \min\{\beta_3/4, 10^{-4}, \eta_{13}/10\}.$$

Choose any β with $0 < \beta < \beta_{13}$. We input 2β to Theorem 3 and get $n_3 = n_3(2\beta, \mu_3, \eta_3, \alpha_0)$.

Finally, define

$$n_{13} = \max\{n_3, (2\beta)^{-1}\}.$$

Suppose we are given a four-partite graph G of order n, $n > n_{13}$, and a partition $V(G) = V_1 \cup V_2 \cup V_3 \cup V_4$ into independent sets that satisfies the conditions in the statement of Theorem 13, i.e., $|V_i| \ge (1/4 - \beta)n$, $1 \le i \le 4$, and $\Delta(\overline{G}) \le \beta n$. Take any two-multicoloring of its edges, say by *red* and *green*.

Now we consider the graph K obtained from G by adding all edges inside the sets V_i. We color those new edges exclusively by *blue* and keep all other edges of K with the same colors they have in G. Notice that now we have a three-multicoloring of an *almost* complete graph on n vertices. In particular

$$\Delta(\overline{K}) \le \beta n$$

implies that K is a $(1-2\beta)$-dense graph. Since $n \ge n_3$ and $2\beta < \beta_3$, we can apply Theorem 3 to K in order to either find a monochromatic matching of size at least $(1/4 + \eta_3)n \ge (1/4 + 5\eta_{13})n$ (edges), or conclude that the coloring of G is of $EC_1(\alpha_0, \alpha_0)$-type, $EC_2(\alpha_0, \alpha_0)$-type, or $EC_3(\mu_3, 0.7, 0.3, (2\beta)^{1/3})$-type.

Note, however, that our coloring of K is not of any of these types. In fact, first note that all color classes defined by these three types of colorings contain a monochromatic bipartite subgraph where each set in the bipartition has order at least $(1 - \max\{\alpha_0, 0.7\mu_3\})n/4 > n/5$ which are $(1 - \max\{\alpha_0, (2\beta)^{1/3}\})$-dense. In particular, those bipartite graphs are at least 19/20-dense. However, the graph induced by the *blue* edges in K does not have this property, being a union of four cliques of order close to $n/4$ with no edges connecting them. Therefore, there must exist a monochromatic connected matching M of size at least $(1/4+5\eta_{13})n$.

Since there exists no *blue* edge from V_i to V_j, where $i \neq j$, every *blue* connected component has order at most $(1/4+3\beta)n$. As $\beta < \eta_{13}$ and M is connected, M cannot be *blue*. Therefore, M is a monochromatic connected matching in the original coloring of G. Assume, without loss of generality, that M is *red*. From this point on, we will return to work on the original multipartite graph G, i.e., we will ignore the *blue* edges. Let C be the (maximal) connected component of G^r containing M. Recall that this means that all edges of C are colored *red* but they are not necessarily exclusively *red*. If C is non-bipartite, we are done. Therefore, we can assume C is bipartite.

Let $V(C)=X \cup Y$ be an arbitrary bipartition of C and let $Z=V \setminus C$. From the definition of C and the choice of X and Y, no edge inside X, inside Y or from Z to $X \cup Y$ is colored *red*. Therefore, they are exclusively colored *green*. Note that $e(M) \geq (1/4+5\eta_{13})n$ implies

$$|Z| \leq \left(\tfrac{1}{2} - 10\eta_{13}\right)n.$$

For $1 \leq i \leq 4$, denote $X_i = V_i \cap X$, $Y_i = V_i \cap Y$, and $Z_i = V_i \cap Z$. Since $|X| \geq e(M) \geq (1/4+5\eta_{13})n$ and $|X_i| \leq |V_i| \leq (1/4+3\beta)n \leq (1/4+3\eta_{13})n$, at least two of the sets $X_i's$ have order at least $2\eta_{13}n > 2\beta n$. By Lemma 16, these two X_i's induce a (*green*) connected graph. Also, all other vertices in X and in Z have at least one neighbor in the union of those two sets. Therefore, $G^g[X \cup Z]$ is connected. Similarly, $G^g[Y \cup Z]$ is connected. So if $Z \neq \emptyset$, then $G^g[X \cup Y \cup Z]$ is connected. In the next cases, we will prove that this (*green*) component is odd and has a large matching, unless many of the sets X_i, Y_i, Z_i are very small, in which case we will prove that the coloring has the desired structure.

Case 1. $|Z| > \eta_{13}n$.

We claim that we can find a large enough odd connected *green* matching. Because $Z \neq \emptyset$, we have that $G^g[X \cup Y \cup Z]$ is connected. To verify that it is not bipartite, we can easily check that it contains a triangle. In fact, we can assume that, without loss of generality, $|Z_1| > \eta_{13}n/4$, which implies $|Z_1| > 2\beta n$. Look at the orders of the sets X_i's and Y_j's. If there is any edge uv in $G^g[X_2, X_3, X_4]$, we can find a common neighbor of u and v in Z_1 and we are done. But we already know that at least one of X_2, X_3, X_4, say X_2, is larger than $2\beta n$. If either X_3 or X_4 is nonempty, we can find an edge in $G^g[X_2, X_3 \cup X_4]$ and we are done. Then we can assume that X_3 and X_4 are empty. Similarly, either we have a triangle or two of the sets Y_2, Y_3, Y_4 are empty, which means that at least one of Y_3 or Y_4 is empty. Call it Y_i ($i=3$ or 4). Notice now that $Z_i = V_i$ and, in particular, $|Z_i| \geq 2\beta n$, so we can find a triangle in $G^g[X_1, X_2, Z_i]$.

Now, we only need to find a large matching in the *green* component. The basic idea is to use Hall's Theorem to find a matching M_1 in $G[Z, X \cup Y]$ that covers all vertices in Z and afterwards use Corollary 19 to prove that there are large matchings M_2 in $V(X) \setminus V(M_1)$ and M_3 in $V(Y) \setminus V(M_1)$. But in order to use Corollary 19 effectively, we want the difference between the largest part in $V(X) \setminus V(M_1)$ and the sum of the others to be small. So, the matching M_1 needs to be chosen with some care.

We select a set $L \subset X \cup Y$ that shall be avoided by M_1. Let L be a subset of $X \cup Y$ of order $4\lfloor 2\eta_{13}n \rfloor$ containing $\lfloor 2\eta_{13}n \rfloor$ vertices from each of two different X_i's and two different Y_i's, and otherwise arbitrary.

We check that Hall's condition works to find a matching M_1, among the (*green*) edges from Z to $(X\cup Y)\setminus L$, that covers all vertices of Z. In fact, a single vertex in Z, without loss of generality in Z_1, has degree at least $|(X\cup Y)\setminus L|-|X_1\cup Y_1|-\beta n>2(1/4+5\eta_{13})n-(8\eta_{13}n)-(1/4+3\beta)n-\beta n>(1/4+\eta_{13})n$. Then, for any $S\subset Z$, denoting by $N(S)$ the set of neighbors of S in $(X\cup Y)\setminus L$, we have: if $|S|<(1/4+\eta_{13})n$, then $|N(S)|\geq|S|$; and if $|S|\geq(1/4+\eta_{13})n$, then S intersects at least two of the sets Z_i's. Let $z',z''\in Z\cap S$ such that z' and z'' belong to different sets Z_i's. In this case we have

$$\begin{aligned}|N(S)| &\geq |N(\{z',z''\})|\geq|(X\cup Y)\setminus L|-2\beta n\\ &> 2(1/4+5\eta_{13})n-(8\eta_{13}n)-2\beta n>(1/2+\eta_{13})n>|Z|\geq|S|.\end{aligned}$$

Therefore, there is a matching M_1 that covers all vertices of Z. Denote $X'=X\setminus V(M_1)$, $X_i'=X_i\setminus V(M_1)$ and assume, without loss of generality, that X_1' is the largest among X_1', X_2', X_3' and X_4'. Let

$$k=\max\{|X_1'|-(|X_2'|+|X_3'|+|X_4'|),0\}.$$

Since $|X_1'|\leq|V_1|\leq(1/4+3\beta)n$ and because at least one of the sets X_2', X_3', X_4' contains $\lfloor 2\eta_{13}n\rfloor$ vertices from L, we have $k\leq(1/4+3\beta-\lfloor 2\eta_{13}\rfloor)n$. By Corollary 19, applied to $G^g[X_1',X_2',X_3',X_4']$ with $m=\beta n$ and $r=4$, there is a matching M_2 that covers all vertices in X' except for at most

$$k+4\beta n\leq(1/4+7\beta-\lfloor 2\eta_{13}\rfloor)n.$$

The analogous statement holds replacing X_i' by Y_i'.

The conclusion is that $M_1\cup M_2\cup M_3$ leaves uncovered at most

$$2(1/4+7\beta-\lfloor 2\eta_{13}\rfloor)n$$

vertices. Therefore

$$|V(M_1)\cup V(M_2)\cup V(M_3)|\geq|V(G)|-(1/2-2\lfloor 2\eta_{13}\rfloor+14\beta)n\geq(1/2+2\eta_{13})n$$

as desired.

Case 2. $|Z|\leq\eta_{13}n$.

We claim that if $|X|>(1/2+2\eta_{13})n$, we can find a large monochromatic odd connected (*green*) matching in $G^g[X]$. In fact, if $|X|>(1/2+2\eta_{13})n$, then at least three of the sets X_i's are larger than $\eta_{13}n>2\beta n$. Therefore, $G^g[X]$ contains a triangle and, in particular, is not bipartite. Also recall that $G^g[X]$ is connected. Finally, we check that Lemma 17 gives us a large matching inside X: since $|X_i|<(1/4+3\beta)n<|X|/2$, for $1\leq i\leq 4$, no X_i can be larger than the sum of the others, so we apply the lemma and conclude that there exists a matching of order at least $|X|-4\beta n>(1/2+\eta_{13})n$, i.e., the orders of X and Y are close to each other.

Now, we can assume that $|X|,|Y|\leq(1/2+2\eta_{13})n$. Since $|Z|\leq\eta_{13}n$, we have $|X|,|Y|\geq(1/2-3\eta_{13})n=(1-6\eta_{13})n/2$. If there is no *green* edge from X to Y, then we have an $EC_B(6\eta_{13},\beta)$ which in particular is an $EC_B(\alpha_1,\alpha_1)$. Now, assume that there is a *green* edge uv from X to Y. Since $G^g[X]$ and $G^g[Y]$ are connected, we conclude that $G^g[X\cup Y]$ is connected. Using Corollary 19 twice, we can find large *green* matchings inside each of X and Y. In fact, as $|X|>(1/2-3\eta_{13})n$ and $\max\{|X_i|:1\leq i\leq 4\}\leq(1/4+3\beta)n$, the difference between the largest $|X_i|$ and the sum of the others is at most $(3\eta_{13}+6\beta)n$. This implies that there is a matching in $G^g[X]$ that misses at most $((3\eta_{13}+6\beta)+4\beta)n$ vertices of X. Similarly, there is a matching in $G^g[Y]$ that misses at most $(3\eta_{13}+10\beta)n$ vertices of Y. The union of those matchings is a (very) large *green* connected matching M: it covers almost all vertices of G and we only need to cover $(1/2+2\eta_{13})n$ vertices.

If either X or Y has at least three nonempty parts, then we can find a triangle, as in the beginning of the previous case, in which case M is an odd matching and we are done. Otherwise, at least two of X_i's and two of Y_i's are empty. We can assume, without loss of generality, that the sets X_3 and X_4 are empty. This implies that $|X_1|,|X_2|\geq((1/4-\beta)-\eta_{13})n\geq(1-5\eta_{13})n/4$. Therefore, $|Y_1|,|Y_2|\leq 5\eta_{13}n$ and, as $|Y|\geq(1/2+2\eta_{13})n$ and $|Y_i|\leq n/4$ for all i, we have that $|Y_3|$ and $|Y_4|$ are non-empty. It follows that Y_1 and Y_2 must be empty, which implies $|Y_3|,|Y_4|\geq(1-5\eta_{13})n/4$.

We are getting closer to prove that the coloring of G must be an $EC_A(5\eta_{13},\beta)$. In fact, we already know that there is no *red* edge in $G[X_1,X_2]$ or $G[Y_3,Y_4]$. We can assume, without loss of generality, that the *green* edge uv from X to Y is such that $u\in X_1$ and $v\in Y_3$. If there is any *green* edge in $G[X_1,Y_4]$ we can greedily construct an odd *green* cycle, in which case M will be odd. Therefore we can assume that there is no *green* edge in $G[X_1,Y_4]$. Similarly, we can assume that there is no *green* edge in $G[X_2,Y_3]$. Then, we conclude that our coloring is of type $EC_A(5\eta_{13},\beta)$ which in particular is an $EC_A(\alpha_1,\alpha_1)$. ■

7. PATHS AND CYCLES IN (BIPARTITE) GRAPHS AND PROOFS OF LEMMAS 14 AND 15

The aim of this section is to prove Lemmas 14 and 15. To this end, we will need the following fact which appears as Theorem 15 of Chapter 10 of Berge [3].

Lemma 20. *Let G be a bipartite graph with the partition $V(G)=A\cup B$ where $|A|=|B|=n\geq 2$. Assume that $\delta(G)\geq 2$ and that for each j, $2\leq j\leq(n+1)/2$, in each of the sets A, B, the number of vertices of degree at most j is smaller than $j-1$. Then G is Hamilton-connected, i.e., each pair of vertices v, w with $v\in A$ and $w\in B$ can be connected by a Hamiltonian path.*

The next easy lemma, originally from [1] (in Portuguese), states that we can find long paths in bipartite graph with large minimum degree. We give a proof here for easy reference.

Lemma 21. *Let H be a bipartite graph with bipartition $X\cup Y$, $|X|,|Y|\geq 4$, and let p and q be integers such that $0\leq p<|X|/3$ and $0\leq q<|Y|/3$. Assume that for every $x\in X$, $\deg(x,Y)\geq|Y|-q$ and for every $y\in Y$, $\deg(y,X)\geq|X|-p$. Then*

- (*a*) *for every two vertices $x,x'\in X$, there exists an (x,x')-path of length $2k-2$ for every $2\leq k\leq\min\{|X|,|Y|-2q\}$; the analogous statement, obtained by exchanging the two vertex classes, also holds;*
- (*b*) *for every two vertices $x\in X$, $y\in Y$, there exists an (x,y)-path of length $2k-1$ for every odd $2\leq k\leq\min\{|X|-2p,|Y|-2q\}$.*

Proof. The idea is just to build paths in a bipartite graph in a greedy fashion. In order to prove (a), we first select k distinct vertices $x_1,\dots,x_k\in X$ (recall $k\leq|X|$) such that $x_1=x$, $x_k=x'$. It is easy to build a path $P_k=x_1y_1x_2y_2\dots y_{k-1}x_k$, with $y_i\in Y$ for all i, $1\le i\le k-1$. Assuming that for a given ℓ, $1\le\ell\le k-1$, we have built $P_\ell=x_1y_1\dots y_{\ell-1}x_\ell$, let y_ℓ be any vertex in the common neighborhood of x_ℓ and $x_{\ell+1}$ which is not in $V(P_\ell)$. Then set $P_{\ell+1}=P_\ell y_\ell x_{\ell+1}$. Such a vertex exists as

$$|(N(x_{\ell-1})\cap N(x_\ell))\setminus V(P_\ell)|\ge(|Y|-2q)-(\ell-1)\ge 2>1$$

since

$$\ell\le k-1\le|Y|-2q-1.$$

The proof of (b) is similar: first take a neighbor x' of y such that $x'\ne x$, and then apply the previous construction to find a path of length $2k$ from x to x', while making sure such that this path also avoids y.

Lemma 22. *Let $r\ge 3$ and let G be an r-partite graph of order $n\ge 3$, with parts V_i such that $|V_i|\le\lfloor n/2\rfloor$, $1\le i\le r$. Assume that each V_i is partitioned into $X_i\cup W_i$ such that $|\bigcup_{i=1}^r W_i|<n/(2r)$ and for every $i\ne j$, the graphs, $G[X_i,X_j]$ and $G[X_i,W_j]$ are complete. Then G has a Hamiltonian cycle.*

Proof. In this proof, contrary to our standard notation, we write P_k for a path with $2k$ vertices. We also set $V_i^k=V_i\setminus V(P_k)$, $X_i^k=X_i\setminus V(P_k)$, $W_i^k=W_i\setminus V(P_k)$, $V^k=\bigcup_{i=1}^r V_i^k$, $W^k=\bigcup_{i=1}^r W_i^k$ and $n_k=|V^k|=n-2k$.

We say that a path P_k in G is *good* if it is such that

(a) $|V_i^k|\le\lfloor n_k/2\rfloor$ for every $1\le i\le r$;
(b) and that either $|W^k|\le 1$ or $|W^k|<n_k/r$ whenever k is odd and $|W^k|<n_k/(2r)$ whenever k is even.

First, we prove by induction on k that, for $k\le\lfloor(n-2)/2\rfloor$, there exists a good path P_k.

For $k=1$, we let $P_k=x_1y_1$, where x_1 is a vertex belonging to a largest class V_i and y_1 a vertex belonging to the second largest class. One can easily check that this is a good path. Now, assume that $P_k=x_kx_{k-1}\dots x_1y_1\dots y_{k-1}y_k$ is a good path for some $k\le\lfloor(n-2)/2\rfloor-1$.

We claim that we can extend P_k to a good path P_{k+1} by adding a new neighbor to each endpoint of P_k. Let i_k be such that $|V_{i_k}^k|$ is maximum among $|V_1^k|,\dots,|V_r^k|$. Select two vertices u, v such that $u\in V_{i_k}^k$, $v\in V^k\setminus V_{i_k}^k$, u is adjacent to one of x_k, y_k and v is adjacent to the other. Notice that $|W^k|<n_k/r$ implies that $X_{i_k}^k=V_{i_k}^k\setminus W^k$ and $X^k\setminus X_{i_k}^k$ are nonempty, therefore we have no trouble with the existence of u and v (even if $x_k,y_k\in W^k$). But we require extra care while choosing v. In the case where $|V_{i_k}^k|=(n_k-1)/2$, two things can happen: either all other classes V_i^k have order strictly less than $(n_k-1)/2$ or there are only three nonempty classes, two of order $(n_k-1)/2$ and one of order 1. In the latter case, we require v to be chosen from the large class not containing u. We also assume that u and v are chosen from W^k whenever this is possible. Finally, we let $\{x_{k+1},y_{k+1}\}=\{u,v\}$ and $P_{k+1}=y_{k+1}y_k\dots y_1x_1\dots x_kx_{k+1}$.

We claim that for the choice of u, v as above the path P_{k+1} is good. The fact that $|V_i^{k+1}|\le\lfloor n_{k+1}/2\rfloor$ is straightforward. One also verify that for every i, with $1\le i\le k$, either $|W^i|\le 1$ or at least one among the vertices x_i,y_i,x_{i+1},y_{i+1} is chosen from W. In fact, if both x_i,y_i are not in W, then x_{i+1} or y_{i+1} can be chosen from W except in the particular case where there are only three nonempty classes, two of order $(n_i-1)/2$ and one of order 1 and in which the only vertex of W is that in the class of order 1. If $k+1$ is even, then the facts that $n_{k+1}=n_{k-1}-4$, one $x_k, y_k, x_{k+1}, y_{k+1}$ is in W and $|W^{k-1}| \leq n_{k-1}/(2r)$ imply that $|W^{k+1}| \leq n_{k-1}/(2r)-1 \leq n_{k+1}/(2r)$. If $k+1$ is odd, the fact that $|W^k| \leq n_k/(2r)$ implies that $|W^{k+1}| \leq n_{k+1}/r$. Therefore, P_{k+1} is good. Next, to prove hamiltonicity, we treat the case whether n is even or n is odd separately.

First, we assume that n is odd. Let $k=(n-3)/2$. We conclude that there exists a good path $P_k = y_k y_{k-1} \ldots y_1 x_1 \ldots x_{k-1} x_k$ (of order $2k=n-3$), such that $P_{k-1} = y_{k-1} \ldots y_1 x_1 \ldots x_{k-1}$ is also good. Let $V^k = V \setminus V(P_k) = \{a,b,c\}$. The fact that P_{k-1} is good implies that at most one of x_k, y_k, a, b, c is in W. And the fact that P_k is good implies that a, b and c belong to different partition classes. Therefore a, b, c are adjacent to each other. Also, two of them, say a,b, are such that a is adjacent to x_k and b is adjacent to y_k. Therefore, we have a Hamiltonian cycle $C_n = cby_{(n-3)/2} \ldots y_1 x_1 \ldots x_{(n-3)/2} ac$.

Finally, assume that n is even. Let $k=(n-2)/2$. As in the previous case, we consider a good path denoted by $P_k = y_k y_{k-1} \ldots y_1 x_1 \ldots x_{k-1} x_k$ (of order $2k=n-2$), and so that P_{k-1} is also good and we let $V^k = V \setminus V(P_k) = \{a,b\}$. Using that P_k and P_{k-1} are good we conclude that at most one among x_k, y_y, a, b is in W and that a and b are in different partition classes. Therefore, we have a Hamiltonian cycle $C_n = by_{(n-2)/2} \ldots y_1 x_1 \ldots x_{(n-2/2} ab$. ■

We restate the Lemma 14 for easy reference.

Lemma 14. *For $n \geq 3$ odd, let $G = K_{(n-1)/2,(n-1)/2,(n-1)/2,(n-1)/2,1}$, let u be its only vertex of degree $2n-2$ and let $H = G \setminus \{u\}$. There exists $\alpha_{14} > 0$ such that, for all $\alpha \leq \alpha_{14}$ and $\delta \leq \alpha$, there is a positive integer n_{14} with the following property: for every odd $n \geq n_{14}$, every two-coloring of G, such that the induced coloring in H is of type $EC_B(\alpha,\delta)$, contains a monochromatic C_n.*

Proof. We set

$$\alpha_{14} = 10^{-4}$$

and consider any $\alpha \leq \alpha_{14}$. Note that, for every $\delta \leq \alpha$, any coloring of type $EC_A(\alpha,\delta)$ is also of type $EC_A(\alpha,\alpha)$, hence, we may assume that $\delta = \alpha$. Take

$$n_{14} = \lfloor \alpha^{-4} \rfloor.$$

Select n odd, with $n \geq n_{14}$. We let $V(G) = U_1 \cup U_2 \cup U_3 \cup U_4 \cup \{u\}$, where U_1, U_2, U_3, U_4 are independent sets of order $(n-1)/2$ and u is the (only) vertex of degree $2n-2$. We also let $H = G \setminus \{u\}$. Consider any two-coloring of G such that the coloring restricted to H is of type $EC_A(\alpha,\alpha)$. We aim to find a monochromatic C_n in this coloring. Let A, B, C, D be sets satisfying conditions (a), (b) and (c) of $EC_A(\alpha,\alpha)$ and notice that we must have $A \subset U_1$, $B \subset U_2$, $C \subset U_3$, $D \subset U_4$ (without loss of generality on the ordering of the sets U_i). Also, let $Z = V(H) \setminus (A \cup B \cup C \cup D)$.

Now, consider the vertex u with full degree and look at the color of the edges from u to $A \cup B \cup C \cup D$.

Claim 23. *If u has red neighbors in both A and B, we can find a monochromatic C_n. Similarly, if u has red neighbors in both C and D or green neighbors in both B and C or green neighbors in both A and D, we can find a monochromatic C_n.*

Proof. Suppose that there exists $a \in A$ and $b \in B$ such that ua and ub are *red*. We show how to find a C_n in this case, and the other cases are analogous.

We show that if there exists a pair of vertex-disjoint *red* edges between $A \setminus \{a\}$ and C, say a_1c_1 and a_2c_2, with $a_i \in A \setminus \{a\}$ and $c_i \in C$, $i = 1, 2$, one can find a *red* C_n. In fact, we can find such a path by applying Lemma 21 a few times with $p = q = \alpha 2n$. More precisely, there exists an (b, a_1)-path P in $G^r[A \setminus \{a\}, B]$ of length 3. Also, there is a (c_1, c_2)-path Q in $G^r[C, D]$ of any even length between 2 and $2(\min\{|C|, |D| - 2\alpha(2n)\}) - 2$, and a (a_2, a)-path R in $G^r[A \setminus V(P), B \setminus V(P)]$ for any even length between 2 and $2\min\{|A \setminus V(P)|, |B \setminus V(P)| - 2\alpha(2n)\} - 2$.

Then, for any even number k between 4 and

$$2(\min\{|A \setminus V(P)|, |B \setminus V(P)| - 2\alpha(2n)\} + \min\{|C|, |D| - 2\alpha(2n)\}) - 4 \tag{5}$$

we can choose Q and R so that $e(Q) + e(R) = k$. Clearly, $P \cup Q \cup R \cup \{au, ub, a_1c_1, a_2c_2\}$ is a copy of C_{k+7}. Notice from the above expression that we can take $k = n - 7$ with room to spare.

In fact, by condition (a) of $EC_A(\alpha, \delta)$ we have

$$|A \setminus V(P)|, |B \setminus V(P)|, |C|, |D| \geq \frac{(1-\alpha)(n-1)}{2} - 2.$$

Together with the bound (5), we have that k can be any even number between 4 and $2((1-\alpha)(n-1) - 8\alpha n) - 4 = 2n - 18\alpha n - 6 + 2\alpha$, which is much bigger than $n - 7$.

This means that we can assume there is no *red* edge in $E(A \setminus \{a\}, C)$, with the exception of at most one *red* star. This implies that all *red* edges in $E(A, C)$ are contained in at most two stars. By the same argument there are no *red* edges in $E(B, D)$ with the exception of at most two *red* stars. So, almost all edges in $E(A \cup B, C \cup D)$ are *green*.

Now, again by Lemma 21 with $p = q = \alpha(2n)$, this time applied to $G^g[A \cup B, C \cup D]$, for any $x, y \in A \cup B$, we can find a (x, y)-path of any given even length between 2 and $2(\min\{|A \cup B|, |C \cup D|\} - 2\alpha(2n)) - 2$. We remark that when $x = a$, we cannot apply the lemma directly (as a might not satisfy the condition $\deg(a, C \cup D) \geq |C \cup D| - \alpha(2n)$), but we still can select one of its *green* neighbors in D, say d, and use the lemma to find a long path from d to y. Again, the upper estimate on the order of our path is close to $2n$ and is clearly larger than $n - 1$. Therefore, if there is any *green* edge xy with $x \in A$ and $y \in B$, we can find a *green* C_n.

Now, we can assume that *all* edges in $G[A, B]$ are *red*. Similarly, we can assume that *all* edges in $G[C, D]$ are *red*. Once more, by applying Lemma 21 to $G^g[A \cup B, C \cup D]$, for any $x \in A \cup B$ and $y \in C \cup D$, we can find a (x, y)-path of any odd length up to almost $2n$ and in particular we can find a (x, y)-path of length $n - 2$. Therefore, if there is any vertex in $Z \cup \{u\}$ which has *green* neighbors in both $A \cup B$ and $C \cup D$ we can find a *green* C_n. Now, we can assume that this does not happen, which means that we can partition the set $Z \cup \{u\}$ into sets S and T such that the vertices in S have only *red* neighbors in $A \cup B$ and the vertices in T have only *red* neighbors in $C \cup D$. Since we have $2n - 1$ vertices in total (in G), either $A \cup B \cup S$ or $C \cup D \cup T$ has at least n vertices. Without loss of generality, we can assume $|A \cup B \cup S| \geq n$. Let W be any subset of S such that $|A \cup B \cup W| = n$.

Notice that now we can apply Lemma 22 to find a *red* C_n in $G[A \cup B \cup W]$ as follows: denote $X_1 = A$, $X_2 = B$, $X_3 = X_4 = X_5 = \emptyset$, $W_i = W \cap U_i$, for $1 \leq i \leq 4$ and $W_5 = W \cap \{u\}$. Clearly, $|X_i \cup W_i| \subset |U_i| \leq \lfloor n/2 \rfloor$ and $|W| \leq |Z \cup \{u\}| \leq \alpha(2n - 2)$, so the conditions of the lemma are satisfied. Therefore, we can find a *red* C_n. This finishes the proof of the claim. ■

Now, select any $a \in A$. From the symmetry of the coloring, we can assume that ua is *red*. Applying Claim 23 repetitively, either we find a C_n, or we can assume that all edges from u to B are *green*, all edges from u to C are *red*, all from u to D are *green* and all from u to A are *red*. Take $b \in B$, $c \in C$, $d \in D$ (so now ua and uc are *red* while ub and ud are *green*). Look at the edges from A to C.

Suppose there is a *red* edge $xy \in E(A,C)$, with $x \neq a$ and $y \neq c$. The same technique from the proof of the claim works here, i.e., we can use Lemma 21 to find an even length (a,x)-path P in $G^r[A,B]$ and an even length (b,y)-path Q in $G^r[C,D]$ so that $P \cup Q \cup \{au, uc, xy\}$ is a *red* C_n. We conclude that most edges in $E(A,C)$ must be *green*. But the technique works in this case as well: take any *green* edge rs, such that $r \in A \setminus \{a\}$ and $s \in C \setminus \{c\}$, and take an odd length (r,d)-path P in $G^g[A,D]$ and an odd length (s,b)-path Q in $G^g[B,C]$ such that $P \cup Q \cup \{rs, bu, ud\}$ is a *green* C_n.

This completes the proof. ■

We restate Lemma 15 for easy reference.

Lemma 15. *For $n \geq 3$ odd, let $G = K_{(n-1)/2,(n-1)/2,(n-1)/2,(n-1)/2,1}$, let u be its only vertex of degree $2n-2$ and let $H = G \setminus \{u\}$. There exists $\alpha_{15} > 0$ such that, for all $\alpha \leq \alpha_{15}$ and $\delta \leq \alpha$, there is a positive integer n_{15} with the following property: for every odd $n \geq n_{15}$, every two-coloring of G, such that the induced coloring in H is of type $EC_B(\alpha, \delta)$, contains a monochromatic C_n.*

Proof. As in the previous proof, we set

$$\alpha_{15} = 10^{-4}$$

and consider any $\alpha \leq \alpha_{15}$. Again, note that for every $\delta \leq \alpha$, any coloring of type $EC_B(\alpha, \delta)$ is also of type $EC_B(\alpha, \alpha)$, hence, we may assume that $\delta = \alpha$. Take

$$n_{15} = \lfloor \alpha^{-4} \rfloor.$$

Let n be odd, with $n \geq n_{15}$. We let $V(G) = U_1 \cup U_2 \cup U_3 \cup U_4 \cup \{u\}$, where U_1, U_2, U_3, U_4 are independent sets of order $(n-1)/2$ and u is the (only) vertex of degree $2n-2$. We also let $H = G \setminus \{u\}$. Consider any two-coloring of G such that the coloring restricted to H is of type $EC_B(\alpha, \alpha)$. We aim to find a monochromatic C_n in this coloring.

Let $X \cup Y \cup Z$ be a partition of $V(H)$ where X and Y satisfies either conditions (a)-(d) of $EC_B(\alpha, \delta)$. Let $X_i = X \cap U_i$, $Y_i = Y \cap U_i$. In particular, $|X|, |Y| \geq (1-\alpha)(n-1)$ which implies that $|Z| \leq \alpha(2n-2)$.

We claim that if there is any *red* edge inside X we can find a *red* C_n. To see that, assume that wx is such an edge. Let y be any *red* neighbor of x in Y. We claim that we can construct a (w,y)-path P of length $n-2$ in $G^r[X \setminus \{x\}, Y]$. We choose subsets $X' \subset X$ and $Y' \subset Y$ such that:

(a) $w \in X'$, $x \notin X'$, $y \in Y'$,
(b) $|X'| = |Y'| = (n-1)/2$, and
(c) $|X'_i \cup Y'_i| \leq (n-1)/4 + \alpha n$, where $X'_i = X' \cap U_i$ and $Y'_i = Y' \cap U_i$.

This can be done because $(1+\alpha)(n-1) \geq |X|, |Y| \geq (1-\alpha)(n-1)$ and $|X_i \cup Y_i| \leq |U_i| = (n-1)/2$. In fact, for example, one can start taking half of the elements of each set X_i and Y_i (rounded to the closest integer), so that property (c) will be true with some room to spare, and then add or subtract at most $\alpha n/2$ vertices to each X' and Y', so that properties (a) and (b) get satisfied.

Let us check that the conditions in Lemma 20 are satisfied for the graph $G^r[X', Y']$. Let $2 \leq j \leq (|Y'|+1)/2$ and write $j = (|Y'|+1)/2 - k = (n+1)/4 - k$, for some $0 \leq k \leq (|Y'|+1)/2 - 2$. Let $R_j = \{v \in X' : \deg(v, Y') \leq j\}$. We need to check that $|R_j| < j-1$. We claim that for every $1 \leq i \leq 4$, either $R_j \cap X'_i = \emptyset$ or $|R_j \cap X'_i| \leq 3\alpha n$. Assume $R_j \cap X'_i \neq \emptyset$ and let $v \in R_j \cap X'_i$, for some $1 \leq i \leq 4$. Since v is adjacent to all but at most $\alpha(2n-2)$ vertices in $\bigcup_{t \neq i} Y_t$, we have that

$$\sum_{t \neq i} |Y'_t| - \alpha(2n-2) \leq \deg(v, Y') \leq j = \frac{|Y'|+1}{2} - k.$$

Therefore

$$|Y'_i| = |Y'| - \left(\sum_{t \neq i} |Y'_t| \right) \geq \frac{|Y'|-1}{2} + k - \alpha(2n-2) = \frac{n-3}{4} + k - 2\alpha n.$$

This and condition (c) above implies that

$$|R_j \cap X'_i| \leq |X'_i| = |X'_i \cup Y'_i| - |Y'_i| \leq 3\alpha n - (k-1) \leq 3\alpha n.$$

We conclude that if $R_j \neq \emptyset$ then $(k-1) \geq 3\alpha n$. Whereas $(k-1) \leq 3\alpha n$, for every i such that $R_j \cap X'_i \neq \emptyset$ we have $|X'_i| \leq 3\alpha n - (k-1) \leq 3\alpha n$.

We conclude that $|R_j| \leq 12\alpha n$. Since $j-1 > (n+1)/4 - (k-1) - \alpha n \geq (n+1)/4 - 4\alpha n \geq 12\alpha n$, we have $|R_j| < j-1$ as claimed. Therefore, we can use Lemma 20 to find a (*red*) Hamiltonian path in $G^g[X', Y']$ starting on w and ending in y. Appending the edges wx and xy to this path we get a *red* C_n.

We can assume now that $G[X]$ has all its edges colored in *green*, i.e., it is a complete *green* multipartite graph. And similarly we conclude that all edges in $G[Y]$ are also *green*. Also, if there is any vertex z in Z such z has a *red* neighbors x, y with $x \in X$ and $y \in Y$, we can use the same argument from above to find a (x,y)-path P in $G^r[X, Y]$ such that $P \cup \{xz, zy\}$ is a (*red*) C_n. Finally, if this does not happen, the set $Z \cup \{u\}$ can be partitioned into $S \cup T$ such that all edges from S to X and all edges from T to Y are *green*. Since the total number of vertices in G is $2n-1$, we have that either $|X \cup S| \geq n$ or $|Y \cup T| \geq n$. Assume, without loss of generality, that the first holds. Letting W be any subset of S such that $|X \cup W| = n$, one can apply Lemma 22 to find a *green* C_n in $G[X \cup W]$. In fact, the conditions of the lemma are satisfied by the sets $V_i = X_i \cup W_i$ where $W_i = W \cap U_i$, for $1 \leq i \leq 4$, $W_5 = W \cap \{u\}$ and $X_5 = \emptyset$. ■

8. FINAL REMARKS

In a recent article, Li et al. [15] conjectured that a generalization of Theorem 2 holds.

Conjecture 24. *Let $N \geq 4$ and let G be a graph of order N with $\delta(G) > 3N/4$. For any 2-coloring of the edges of G and any k, $4 \leq k \leq \lceil N/2 \rceil$, G contains a monochromatic C_k.*

In [15], it is proved that, for any $\varepsilon > 0$ and n large, the same assumptions imply that we can find a monochromatic C_k for every k between 4 and $\lfloor (1/8-\varepsilon)N \rfloor$.

In Theorem 2, letting $N=2n-1$, we have that all vertices, except one, that we called u, have degree $\lceil 3N/4 \rceil$. One can adapt our proof to the case where u does not have full degree, but has only degree $\lceil 3N/4 \rceil$. In fact, in order to do that, one only need to change the proofs of Lemmas 14 and 15. Though we have only searched for a $C_{\lceil N/2 \rceil}$, this provides a tight example in which the above conjecture is probably true. We hope to attack the general problem as well as similar problems about even cycles and paths in forthcoming articles.

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