Connected greedy colourings

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Abstract. A *connected vertex ordering* of a graph G is an ordering $v_1 < v_2 < \cdots < v_n$ of $V(G)$ such that v_i has at least one neighbour in $\{v_1, \ldots, v_{i-1}\}$, for every $i \in \{2, \ldots, n\}$. A *connected greedy colouring* is a colouring obtained by the greedy algorithm applied to a connected vertex ordering. In this paper we study the parameter $\Gamma_c(G)$, which is the maximum k such that G admits a connected greedy k-colouring, and $\chi_c(G)$, which is the minimum k such that a connected greedy k-colouring of G exists. We prove that computing $\Gamma_c(G)$ is NP-hard for chordal graphs and complements of bipartite graphs. We also prove that if G is bipartite, $\Gamma_c(G) = 2$. Concerning $\chi_c(G)$, we first show that there is a k-chromatic graph G_k with $\chi_c(G_k) > \chi(G_k)$, for every $k \geq 3$. We then prove that for every graph G, $\chi_c(G) \leq \chi(G) + 1$. Finally, we prove that deciding if $\chi_c(G) = \chi(G)$, given a graph G, is a NP-hard problem.

Keywords: Vertex colouring, Greedy colouring, Connected greedy colouring

1 Introduction

A *k-colouring* of a graph $G = (V, E)$ is a surjective mapping $\psi : V \to \{1, 2, \ldots, k\}$ such that $\psi(u) \neq \psi(v)$ for any edge $uv \in E$. A *k-colouring* may also be seen as a partition of the vertex set of G into k disjoint *stable sets* $S_i = \{v \mid \psi(v) = i\}$, $1 \leq i \leq k$. The elements of $\{1, \ldots, k\}$ are called *colours*, and the set of vertices with a given colour is a *colour class*. A graph is *k-colourable* if it admits a *k-colouring*. The minimum number of colours in a colouring of a graph G is its *chromatic number*, defined as $\chi(G) = min\{k \mid G \text{ is } k\text{-colourable}\}$. We say that G is k-chromatic if $\chi(G) = k$.

Graph colourings are a natural model for problems in which a set of objects is to be partitioned according to some prescribed rules. For example, problems of *scheduling* [11], *frequency assignment* [5], *register allocation* [2,3], and the *finite element*

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method [9], are naturally modelled by colourings. While it is easy to find a colouring when no bound is imposed on the number of colour classes, for most of these applications the challenge consists in finding one that minimizes the number of colours.

To decide if a graph admits a colouring with k colours is an NP-complete problem, even if k is not part of the input [7]. Moreover, the chromatic number is hard to approximate: for all $\epsilon > 0$, there is no algorithm that approximates the chromatic number within a factor of $n^{1-\epsilon}$ unless P = NP [8,13].

Greedy colourings and its best and worst case behaviour. The most basic and widespread algorithm producing colourings is the *greedy algorithm* or *first-fit algorithm*. Given a vertex ordering $\sigma = v_1 < v_2 < \cdots < v_n$ of $V(G)$, the greedy algorithm colours the vertices in the order σ assigning to v_i the smallest positive integer not already used in its lower-indexed neighbours. A *greedy colouring* is a colouring obtained from the greedy algorithm.

A remarkable property of the greedy algorithm is that it is always possible to find an optimal colouring by using it. That is, given any graph G, there exists an ordering of $V(G)$ such that the greedy algorithm produces a greedy colouring with $\chi(G)$ colours. To see that this is true, consider a colouring $S_1, S_2, \ldots, S_k$ and any vertex ordering in which the vertices of S_i precede those of S_{i+1}, for $1 \leq i \leq k-1$. The greedy algorithm applied to any such ordering produces a greedy colouring with at most k colours. When choosing $k = \chi(G)$, we get a greedy colouring with $\chi(G)$ colours.

Although greedy colourings with an optimal number of colours exist, this property is not achieved by any vertex ordering. Consider for example the path on four vertices P_4. Any ordering of the vertices of P_4 in which the vertices of degree one precede the vertices of degree two produces a greedy colouring with three colours. The worst-case behaviour of the greedy algorithm on a graph G is measured by the *Grundy number* $\Gamma(G)$, which is the largest k such that G has a greedy k-colouring. It's known that $\chi(G) \leq \Gamma(G) \leq \Delta(G) + 1$. Unfortunately, colourings obtained by the greedy algorithm can be arbitrarily far from an optimal colouring. The difference $\Gamma(G) - \chi(G)$ can be arbitrarily large, even for trees. This can be seen with the *k-binomial-tree* $\mathcal{B}_k$, first defined in [1]. The tree $\mathcal{B}_1$ is the tree on one vertex. The tree $\mathcal{B}_k$, for $k \geq 2$, is built from a copy of $\mathcal{B}_{k-1}$ by adding $|V(\mathcal{B}_{k-1})|$ new vertices and matching them with the vertices from the copy of $\mathcal{B}_{k-1}$. A simple induction can be used to show that $\Gamma(\mathcal{B}_k) = k$, while in fact $\chi(\mathcal{B}_k) = 2$, since $\mathcal{B}_k$ is a tree.

While the Grundy number can be computed in polynomial time for trees [1] and partial k-trees [10], the corresponding optimization problem is NP-hard for general graphs. It remains NP-hard for complements of bipartite graphs [12], bipartite graphs and chordal graphs [6].

Connected greedy colourings. A *connected greedy colouring* of a connected graph $G = (V, E)$ is a greedy colouring obtained from a *connected ordering* $\sigma = v_1 < v_2 < \cdots < v_n$ of $V(G)$, that is, an ordering of the vertices with the property that v_i has at least one neighbour in $\{v_1, \ldots, v_{i-1}\}$ for every $i \in \{2, \ldots, n\}$. In other words, if $V_i = \{v_1, \ldots, v_i\}$, then $G[V_i]$ is connected for every $i \in \{2, \ldots, n\}$. In this paper we study connected greedy colourings with a focus on upper and lower bounds for the number of colours used.

The paper is organized as follows. In Section 2 we consider the worst-case behaviour of connected greedy colourings. In order to do so we define the connected Grundy number $\Gamma_c(G)$, which is the maximum k such that G admits a connected greedy colouring with k colours. We prove that, for a bipartite graph, the connected Grundy number is always equal to the chromatic number of the graph. We show that the difference $\Gamma_c(G) - \chi(G)$ can be arbitrarily large for chordal planar graphs. We also show that determining the connected Grundy number is NP-hard on chordal graphs and complements of bipartite graphs. In Section 3 we prove that, in contrast to what happens with greedy colourings, there are graphs G for which there is no connected greedy colourings with $\chi(G)$ colours. Motivated by this fact, we define $\chi_c(G)$ as the smallest k such that the graph admits a connected greedy colouring with k colours. We prove that $\chi_c(G) \leq \chi(G)+1$, for any graph G. We then show that, given a graph G, deciding if $\chi_c(G) = \chi(G)$ is a NP-hard problem.

2 The worst-case behaviour

In order to analyse the worst-case behaviour of connected greedy colourings, we define an analogue of the Grundy number. The *connected Grundy number* of a graph G, denoted $\Gamma_c(G)$, is the maximum k such that G admits a connected greedy k-colouring. Clearly, $\Gamma_c(G) \leq \Gamma(G)$. The connected greedy algorithm can therefore be seen as an improved version of the greedy algorithm. Indeed, in contrast to what happens with the Grundy number, the connected greedy algorithm always finds an optimal colouring if the input graph is bipartite.

Lemma 1. *Let $G = (A \cup B, E)$ be a connected bipartite graph with at least one edge. Then, $\Gamma_c(G) = 2$.*

Proof. Let $v_1 < v_2 < \cdots < v_n$ be a connected ordering and ψ be the corresponding greedy colouring. Without loss of generality, suppose $v_1 \in A$. We prove by induction on the number of coloured vertices that all coloured vertices in A are coloured 1 and in B are coloured 2. This is true if no vertices are coloured. Now we consider what happens when colouring v_i, for $1 \leq i \leq k$. If $v_i \in A$, then any coloured neighbour of v_i is in B and coloured 2. Therefore, $\psi(v_i) = 1$. If $v_i \in B$, then $i \neq 1$ and any coloured neighbour of v_i is in A and coloured 1. Furthermore, since the ordering is connected, at least one of its neighbours is already coloured so $\psi(v_i) = 2$. □

On the other hand, planar graphs and chordal graphs are examples of graph classes which have connected greedy colourings arbitrarily far from optimal. Before we prove these results, we need the following auxiliary results. If graphs G and H are vertex disjoint, let the *join* $G \vee H$ of G and H be the graph obtained from a copy of G, a copy of H and adding all possible edges with one endpoint in G and another in H. Say that a graph G is *null* if $V(G) = \emptyset$ and *non-null* otherwise.

Lemma 2. *Let G and H be vertex disjoint non-null graphs. Also let ψ_G and ψ_H be greedy colourings of G and H with k_G and k_H colours, respectively. Then there is a connected greedy colouring of the join graph $G \vee H$ that uses $k = k_G + k_H$ colours.*

Proof. Let $\sigma_G = v_1 < v_2 < \cdots < v_p$ and $\sigma_H = u_1 < u_2 < \cdots < u_q$ be the orderings of $V(G)$ and $V(H)$ such that the greedy algorithm produces the colourings ψ_G and ψ_H, respectively. Moreover, let $\sigma_{G\vee H} = v_1 < u_1 < u_2 < \cdots < u_q < v_2 < \cdots < v_p$ be a vertex ordering of $V(G \vee H)$. Since all vertices in G are adjacent to all vertices in H in the graph $G \vee H$, $\sigma_{G\vee H}$ is a connected order. The greedy algorithm applied to $\sigma_{G\vee H}$ first colours v_1 with colour 1, and then colours the vertices from $V(H)$ with colours $\{2, \ldots, k_H + 1\}$, assigning to $u \in V(H)$ the colour $\psi_H(u) + 1$. Now the vertices $v_2, v_3, \ldots, v_q$ all have neighbours with colours from $\{2, \ldots, k_H + 1\}$. Therefore, any vertex $v \in V(G)$ will be coloured 1 if $\psi_G(v) = 1$ and coloured $k_H + \psi_G(v)$ otherwise. □

Corollary 1. *If G and H are disjoint non-null graphs, then $\Gamma_c(G \vee H) = \Gamma(G) + \Gamma(H)$.*

Proof. First note that $\Gamma_c(G \vee H) \leq \Gamma(G \vee H) = \Gamma(G) + \Gamma(H)$. Now, given greedy colourings of G and H with $\Gamma(G)$ and $\Gamma(H)$ colours, respectively, Lemma 2 states that there is a connected greedy colouring of $G \vee H$ with $\Gamma(G) + \Gamma(H)$ colours. Therefore, $\Gamma_c(G \vee H) \geq \Gamma(G) + \Gamma(H)$ which completes the result. □

Let K_n denote the complete graph on n vertices.

Proposition 1. *For every $M \geq 0$, there is a chordal planar graph G such that $\Gamma_c(G) - \chi(G) = M$.*

Proof. Consider a copy H of the binomial tree $\mathcal{B}_{M+2}$. Clearly, H is planar, as it is a tree. Every tree is *outerplanar*, meaning it admits a drawing in which every vertex is in the outer face. Therefore the graph $H' = H \vee K_1$ is also planar. Furthermore, any cycle in H' must use the unique vertex v in K_1. Therefore, H' is also chordal since v is adjacent to all vertices in H. Moreover, we have $\chi(H') = 3$. Now, Corollary 1 tells us that $\Gamma_c(H') = M + 3$. □

Now we consider the computational complexity of determining the connected Grundy number of a graph. We say that a family of graphs $\mathcal{G}$ is *closed under universal vertices* if, given a graph $G \in \mathcal{G}$, the graph $G' = G \vee K_1$ also belongs to $\mathcal{G}$.

Proposition 2. *Let $\mathcal{G}$ be a family of graphs closed under universal vertices such that, given $G \in \mathcal{G}$ and an integer k, the problem of deciding if $\Gamma(G) \geq k$ is NP-complete. Then the problem of deciding if $\Gamma_c(G) \geq k$, given $G \in \mathcal{G}$ and an integer k, is also NP-complete.*

Proof. Let $G \in \mathcal{G}$ and $k \in \mathbb{N}$. Let G' be the graph $G \vee K_1$. From Corollary 1, we have that $\Gamma_c(G') = \Gamma(G) + 1$. Therefore, $\Gamma(G) \geq k$ if and only if $\Gamma_c(G') \geq k + 1$. □

Since chordal graphs and complements of bipartite graphs are graph classes that are closed under universal vertices, and because of the NP-completeness results that were mentioned before, the following result is immediate.

Theorem 1. *Given a graph G and an integer k, deciding if $\Gamma_c(G) \geq k$ is a NP-complete problem. The problem remains NP-complete even if the graph G is restricted to chordal graphs or complements of bipartite graphs.*

3 The best-case behaviour

As previously mentioned, for every graph G there is a greedy colouring of G using $\chi(G)$ colours. In this section, we prove that the same is not true when considering connected greedy colourings. More precisely, we prove the following theorem.

Theorem 2. *For every $k \geq 3$, there is a k-chromatic graph H_k with no connected greedy colouring with k colours.*

Thus, it makes sense to define the minimum number of colours $\chi_c(G)$ in a connected greedy colouring of G. A natural question would be to ask if $\chi_c(G)$ is bounded by a function of $\chi(G)$. We prove such a function exists and that, in fact, $\chi_c(G)$ is bounded by $\chi(G) + 1$.

Theorem 3. *For any connected graph G, we have $\chi_c(G) \leq \chi(G) + 1$.*

Consider the graph G_k, $k \geq 3$, depicted in Figure 3.

Let X_k and Z_k denote the vertex sets of the copies of K_{k-1} adjacent to only $\{a, b\}$ and to only $\{c, d\}$ respectively, and let Y_k denote the vertex set of the remaining copy of K_{k-1}, which is adjacent to all four vertices.

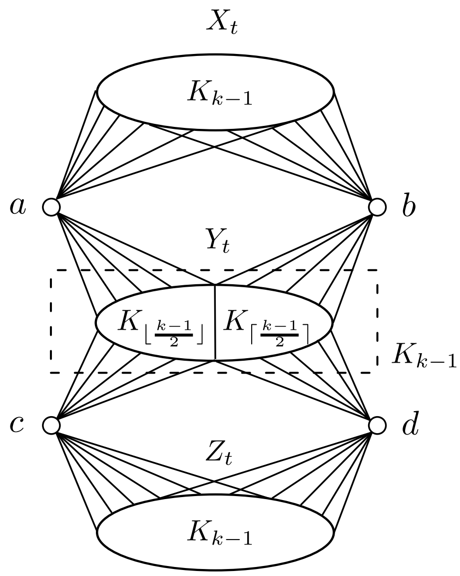

Fig. 1. The graph G_k.

Since in G_k, $X_k \cup \{a\}$ is a clique on k vertices, we have $\chi(G_k) \geq k$. To see that $\chi(G_k) = k$, consider the following k-colouring of G_k. Arbitrarily colour the vertices in X_k, Y_k and Z_k with colours in the set $\{1, \ldots, k-1\}$ and colour a, b, c and d with colour k. Indeed, in the following result we prove that a, b, c and d must always have the same colour in any k-colouring of G_k.

Lemma 3. *Let ψ be a k-colouring of G_k. Then, $\psi(a) = \psi(b) = \psi(c) = \psi(d)$.*

Proof. In any k-colouring ψ of G_k, the vertices of X_k have $k - 1$ distinct colours. Therefore $\psi(a) = \psi(b)$, for otherwise we would need more than k colours. With a similar argument on Z_k we have $\psi(c) = \psi(d)$. Since Y_k induces a clique on $k - 1$ vertices, it should be coloured with $k - 1$ distinct colours. They should be different from $\psi(a)$, since all vertices are adjacent either to a or b, and $\phi(a) = \phi(b)$. Therefore, since the vertices in Y_k are also adjacent either to c or d, we get that $\psi(a) = \psi(b) = \psi(c) = \psi(d)$. □

Let v be a vertex of G and α be a colour. A *(v, α)-connected greedy colouring* of G is a colouring obtained from a connected ordering that starts from v by colouring v with colour α and then colouring the remaining vertices with the greedy algorithm.

Lemma 4. *Let $v \in V(G_k)$ and α be a colour in $\{1, \ldots, k\}$. In any (v, α)-connected greedy k-colouring of G_k, the vertices a, b, c and d have a colour at most $\lceil \frac{k-1}{2} \rceil + 1$.*

Proof. Consider a (v, α)-connected greedy k-colouring of G and, by symmetry, say that $v \in X_k \cup Y_k \cup \{a, b\}$. Since we follow a connected ordering and $\{c, d\}$ is a vertex cut, no vertex from Z_k is coloured before at least one of $\{c, d\}$ is coloured. Let z be the first vertex in the set $\{c, d\}$ that is coloured. Then the only neighbours of z that may have already been coloured are the ones from Y_k, and therefore z has at most $\lceil \frac{k-1}{2} \rceil$ coloured neighbours. Therefore, the colour of z is at most $\lceil \frac{k-1}{2} \rceil + 1$, and from Lemma 3 we get that a, b, c and d get the same colour which is at most $\lceil \frac{k-1}{2} \rceil + 1$. □

In particular, Lemma 4 implies that a colouring of G_k obtained by giving vertex a (alt. b, c or d) a colour greater than $\lceil \frac{k-1}{2} \rceil + 1$ and extending that colouring in a connected greedy way will always use more than k colours.

We are ready to prove Theorem 2.

Proof (of Theorem 2). Let H_k be the graph obtained as follows. Take $\lceil \frac{k-1}{2} \rceil + 2$ copies of G_k and add edges joining all copies of a, thus forming a $(\lceil \frac{k-1}{2} \rceil + 2)$-clique K with these vertices. Since all copies of a are cut vertices separating the copies of G_k from the clique K, we can paste colourings of G_k with each copy of a receiving a different colour to colour H_k with k colours. Furthermore, since H_k contains at least one copy of G_k we have $\chi(H_k) = k$.

Assume to the contrary that σ is a connected ordering of $V(H_k)$ such that the greedy algorithm gives a k-colouring ψ of H_k. Since K forms a clique, there is at least one copy of vertex a with a colour α at least $\lceil \frac{k-1}{2} \rceil + 2$ and call this vertex w. Let v be the first vertex in the connected order σ and let W be the set of vertices corresponding to the copy of G_k to which w belongs. Furthermore, let σ_W be σ restricted to the vertices in W and ψ_W be the colouring of G_k obtained from the vertices in W. Since w is a cut vertex in H_k, then σ_W is connected. Therefore, if $v \in W$, then ψ_W is a $(v, 1)$-connected k-colouring of G_k which colours w with colour α. If $v \notin W$, then ψ_W is a (w, α)-connected k-colouring of G_k. In either case, we get a contradiction to Lemma 4. □

We now show that $\chi_c(G)$ is never greater than $\chi(G)+1$, for any graph G.

Lemma 5. *Let G be a connected graph and v a vertex such that $G-v$ is k-colourable. For any positive integer α, there is a (v,α)-connected greedy colouring such that no vertex in $G-v$ get a colour larger than $k+1$.*

Proof. Since $G-v$ is k-colourable, let $S_1,\dots,S_k$ be a partition of $V(G-v)$ into k stable sets. By induction on k, we prove the stronger assumption that there is a (v,α)-connected greedy colouring such that no vertex in S_i gets a colour larger than $i+1$, for $1\le i\le k$. The result is valid when $k=0$ as $G-v$ is null.

Assume $k\ge 1$. We give an algorithm to obtain the desired colouring. To do so, let $H=G-v-S_k$ and let $\mathcal{C}$ be the set of connected components of H. Start by colouring v with α. We then break the colouring procedure into three phases. In the first phase, we only colour vertices if $\alpha=k+1$. In the second phase, we colour any uncoloured component of $\mathcal{C}$ that contains a neighbour of v. In the third phase, we colour the remaining vertices.

Phase 1. If $\alpha=k+1$, then let $W=N_G(v)\cap S_k$. If W is not empty, we proceed as follows. Start by colouring all vertices in W. Since $\alpha\ge 2$, then all vertices in W are coloured 1. Let G' be the graph obtained from H by adding a vertex w adjacent to any vertex adjacent to W in $G-v$, i.e, $N_{G'}(w)=N_{G-v}(W)$. Let C be the component of G' that contains w and note that $(S_1\cap V(C)),\dots,(S_{k-1}\cap V(C))$ is a partition of $C-w$ into $k-1$ stable sets. By the induction hypothesis, there is a $(w,1)$-connected greedy colouring ψ_C of C such that no vertex in S_i gets a colour larger than $i+1$, for $1\le i\le k-1$. We claim that colouring the vertices in $V(C)-\{w\}$ according to ψ_C is a (v,α)-connected colouring of $G[(V(C)-\{w\})\cup W\cup\{v\}]$. Indeed, any vertex in C is adjacent to w if, and only if, it is also adjacent to a vertex of W in G. Furthermore, any vertex z in C is coloured with a colour no larger than k and, therefore, z is coloured with colour $\psi_C(z)$ by the greedy algorithm even if z is adjacent to v, as v is coloured $\alpha=k+1$.

At the end of Phase 1 we have the following property which is maintained until we end our colouring algorithm: any uncoloured vertex in S_k has no neighbour coloured $k+1$. Indeed, all neighbours of v have been coloured if $\alpha=k+1$, no two vertices in S_k are adjacent as S_k is stable and vertices in S_i, for $i<k$, get colours at most k. Also note that any component in $\mathcal{C}$ is either fully coloured or contains no coloured vertices. Furthermore, any component of $\mathcal{C}$ that is uncoloured has no neighbour in W.

Phase 2. Let $\mathcal{C}_v\subseteq\mathcal{C}$ be the set of uncoloured components of $\mathcal{C}$ that contain a neighbour of v. By the properties obtained at the end of Phase 1, these components have no coloured vertices and no coloured neighbour in G other than v. Let $V_{\mathcal{C}_v}=\bigcup_{C\in\mathcal{C}_v}V(C)$. Let $\hat{G}$ be the subgraph of G induced by $V_{\mathcal{C}_v}\cup\{v\}$ and note that $(S_1\cap V_{\mathcal{C}_v}),\dots,(S_{k-1}\cap V_{\mathcal{C}_v})$ is a partition of $\hat{G}-v$ into $k-1$ stable sets. By the induction hypothesis, there is a (v,α)-connected greedy colouring $\psi_{\mathcal{C}_v}$ of $\hat{G}$ such that no vertex in S_i gets a colour larger than $i+1$, for $1\le i\le k-1$. We colour the vertices in $V_{\mathcal{C}_v}$ according to $\psi_{\mathcal{C}_v}$.

At the end of Phase 2, we maintain the property that any component in $\mathcal{C}$ is either fully coloured or has no coloured vertex. Furthermore, no vertex in any uncoloured component of $\mathcal{C}$ has any coloured neighbour.

Phase 3. In Phase 3, if G has any uncoloured vertices, then we colour the remaining vertices in a sequence of steps. At each step, we colour one vertex w in S_k and all components of $\mathcal{C}$ that contain at least one neighbour of w. At the end of each step, we maintain the property that each component in $\mathcal{C}$ is either fully coloured or contains no coloured vertex. Furthermore, no vertex in any uncoloured component of $\mathcal{C}$ has any coloured neighbour. Note that this is true initially as observed at the end of Phase 2.

The structure of each step is as follows. If G has any uncoloured vertex, then there exists an uncoloured vertex $w \in S_k$ adjacent to at least one coloured vertex. Indeed this must be the case as G is connected and no uncoloured component of $\mathcal{C}$ contains any coloured neighbour. Colour w greedily and let this colour be β. Since w has no neighbour with colour $k+1$, then $\beta \leq k+1$. From here, we follow a structure similar to what was done in Phase 2. Let $\mathcal{C}_w \subseteq \mathcal{C}$ be the set of uncoloured components of $\mathcal{C}$ that contain a neighbour of w. By the properties obtained at the end of Phase 2 and between steps, these components have no coloured vertices and no coloured neighbour in G other than w. Let $V_{\mathcal{C}_w} = \bigcup_{C \in \mathcal{C}_w} V(C)$. Let G'' be the subgraph of G induced by $V_{\mathcal{C}_w} \cup \{w\}$ and note that $(S_1 \cap V_{\mathcal{C}_w}), \ldots, (S_{k-1} \cap V_{\mathcal{C}_w})$ is a partition of $G'' - w$ into $k-1$ stable sets. By the induction hypothesis, there is a (w, β)-connected greedy colouring $\psi_{\mathcal{C}_w}$ of G'' such that no vertex in S_i gets a colour larger than $i+1$, for $1 \leq i \leq k-1$. We colour the vertices in $V_{\mathcal{C}_w}$ according to $\psi_{\mathcal{C}_w}$.

Since we coloured all vertices in components of $\mathcal{C}$ that contain a neighbour of w, the desired property between steps is maintained. Therefore, this algorithm continues until all vertices in G are coloured obtaining the desired colouring. □

With Lemma 5, proving Theorem 3 is simple.

Proof (of Theorem 3). Let G be any connected k-chromatic graph and let v be any vertex of G. Since $G - v$ is k-colourable, we can apply Lemma 5 to obtain a $(v, 1)$-connected greedy colouring of G such that no vertex of $G - v$ gets a colour larger than $k+1$. Since this colouring starts by colouring v with colour 1, this is a connected greedy colouring of G and no vertex has colour larger than $k+1$. □

A natural question that arises is the computational complexity of deciding if $\chi_c(G) = \chi(G)$, given a connected graph G.

Theorem 4. *Let G be a connected graph. To decide if $\chi_c(G) = \chi(G)$ is a NP-hard problem.*

Proof. Consider the k-COLOURABILITY problem, in which the input is a graph G and the question is whether $\chi(G) \leq k$. 3-COLOURABILITY restricted to 4-regular graphs is NP-hard [4]. To see that it is also NP-hard for $k > 3$, observe that if v is a universal vertex, $\chi(G) = \chi(G - v) + 1$, and therefore an instance of $(k-1)$-COLOURABILITY can be reduced to one of k-COLOURABILITY by adding a universal vertex. As a consequence of this reduction, k-COLOURABILITY is NP-hard for $k \geq 4$, even if the input graph G has a universal vertex and $\chi(G) < 2k$. Let G be a graph with these properties. Let H be the graph obtained from G as follows. For every $v \in V(G)$, add a copy G^v of G_{2k-1} and identify v with the copy of vertex a. Then, since $\chi(G_{2k-1}) = 2k-1$ and $\chi(G) < 2k$, we get that $\chi(H) = 2k-1$. We now prove that $\chi_c(H) = \chi(H)$ if and only if $\chi(G) \leq k$.

Suppose $\chi_c(H) = \chi(H)$ and let c be a greedy connected colouring of H with $\chi(H)$ colours. Any vertex $v \in V(G)$ is coloured at most k, since otherwise there is a connected greedy colouring of G_{2k-1} in which vertex a has a colour in $\{k+1, \ldots, 2k-1\}$, contradicting Lemma 4. The restriction of c to the copy of G in H is a colouring with at most k colours, implying $\chi(G) \leq k$.

Suppose now that $\chi(G) \leq k$. In this case, there is a greedy colouring of G that uses at most k colours. Since G has a universal vertex, this greedy colouring can be made a connected colouring, by rearranging the colour classes so that the universal vertex receives colour 1. Let c be the partial colouring of H in which the vertices from G are coloured according to the previous colouring. For any vertex $v \in V(G)$, since its colour is at most k, we may colour the vertices in G^v using only colours smaller than $2k-1$ and while keeping the colouring connected. In this way we obtain a greedy connected colouring of H that uses no colour larger than $2k-1$. Since $\chi(G_{2k-1}) = 2k-1$, we have that $\chi(H) \geq 2k-1$, and therefore $\chi_c(H) = \chi(H) = 2k-1$. □

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