The maximum time of 2-neighbour bootstrap percolation: algorithmic aspects

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Abstract. In 2-neighbourhood bootstrap percolation on a graph G, an infection spreads according to the following deterministic rule: infected vertices of G remain infected forever and in consecutive rounds healthy vertices with at least 2 already infected neighbours become infected. Percolation occurs if eventually every vertex is infected. In this paper, we are interested in calculating the maximum time $t(G)$ the process can take, in terms of the number of rounds needed to eventually infect the entire vertex set. We prove that the problem of deciding if $t(G) \geq k$ is NP-complete for: (a) fixed $k \geq 4$; (b) bipartite graphs with fixed $k \geq 7$; and (c) planar bipartite graphs. Moreover, we obtain polynomial time algorithms for (a) $k \leq 2$, (b) chordal graphs and (c) $(q, q-4)$-graphs, for every fixed q.

1 Introduction

Under r-neighbour bootstrap percolation on a graph G, the spreading rule is a threshold rule in which S_{i+1} is obtained from S_i by adding to it the vertices of G which have at least r neighbours in S_i. We say that a set S_0 percolates G (or that percolation occurs) if eventually every vertex of G becomes infected, that is, there exists a t such that $S_t = V(G)$. In that case we define $t_r(S)$ as the minimum t such that $S_t = V(G)$. And define, the *percolation time of* G as $t_r(G) = \max\{t_r(S) : S \text{ percolates } G\}$. In this paper, we shall focus on the case where $r = 2$ and in such case we omit the subscript of the functions $t_r(S)$ and $t_r(G)$.

Bootstrap percolation was introduced by Chalupa et al. [8] as a model for interacting particle systems in physics. Since then it has found applications in clustering phenomena, sandpiles, and many other areas of sta-

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tistical physics, as well as in neural networks and computer science [10].

There are two broad classes of questions one can ask about bootstrap percolation. The first, and the most extensively studied, is what happens when the initial configuration S_0 is chosen randomly under some probability distribution? One would like to know how likely percolation is to occur, and if it does occur, how long it takes. The answer to the first of these questions is now well understood for various graphs [1,2,15].

The second broad class of questions is the one of extremal questions. For example, what is the smallest or largest size of a percolating set with a given property? Interesting cases are solved in [3–5,7,9,17–19].

Here, we consider the decision version of the problem, as stated below.

PERCOLATION TIME PROBLEM
Input: A graph G and an integer k.
Question: Is $t(G) \geq k$?

It is interesting to notice that infection problems appear in the literature under many different names and were studied by researches of various fields. The particular case in which $r = 2$ in r-neighbourhood bootstrap percolation is also a particular case of a infection problem related to convexities in graph, which are also of our interest.

A finite *convexity space* is a pair $(V, \mathcal{C})$ consisting of a finite ground set V and a set $\mathcal{C}$ of subsets of V satisfying $\emptyset, V \in \mathcal{C}$ and $\mathcal{C}$ is closed under intersection. The members of $\mathcal{C}$ are called *$\mathcal{C}$-convex sets* and the *convex hull* of a set S is the minimum convex set $H(S) \in \mathcal{C}$ containing S.

A convexity space $(V, \mathcal{C})$ is an *interval convexity* [6] if there is a so-called *interval function* $I : \frac{V}{2} \to 2^V$ such that a subset C of V belongs to $\mathcal{C}$ if and only if $I(\{x, y\}) \subseteq C$ for every two distinct elements x and y of C. With no risk of confusion, for any $S \subset V$, we also denote by $I(S)$ the union of S with $\bigcup_{x,y \in S} I(\{x, y\})$. In interval convexities, the convex hull of a set S can be computed by exhaustively applying the corresponding interval function until obtaining a convex set.

The most studied graph convexities defined by interval functions are those in which $I(\{x, y\})$ is the union of paths between x and y with some particular property. Some common examples are the P_3-convexity [12], geodetic convexity [13] and monophonic convexity [11]. We observe that the spreading rule in 2-neighbours bootstrap percolation is equivalent to $S_{i+1} = I(S_i)$ where I is the interval function which defines the P_3-convexity: $I(S)$ contains S and every vertex belonging to some path of 3 vertices whose extreme vertices are in S. It will be convenient to denote S_i by $I_i(S)$, where $I_i(S)$ is obtained by applying i times the operation I. Related to the geodetic convexity, there exists the *geodetic iteration number of a graph* [14] which is similar to our definition of $t(G)$.

2 Results

We first prove the following NP-hardness results.

Theorem 2.1. PERCOLATION TIME *is NP-complete for any fixed* $k \geq 4$. *If the graph is bipartite, it is NP-Complete for any fixed* $k \geq 7$. *It is also NP-Complete for planar bipartite graphs.*

We first make a reduction from 3-SAT. We construct a graph G as follows. For each clause C_i with literals $\ell_{i,1}$, $\ell_{i,2}$ and $\ell_{i,3}$, add to G a gadget as in Figure 2(a). For each pair of literals $\ell_{i,a}$, $\ell_{j,b}$ such that one is the negation of the other, add a vertex $y_{(i,a),(j,b)}$ adjacent to $w_{i,a}$ and $w_{j,b}$. Let Y be the set of all vertices created this way. Finally, add a vertex z adjacent to all vertices in Y and a pendant vertex z' adjacent to z. It is possible to prove that the formula is satisfiable if and only if $t(G) \geq 4$. In the case of bipartite graphs, the reduction is from 4-SAT and the construction is similar, but using the gadget in Figure 2(b). The case of planar bipartite graphs is more technical and is by a reduction from PLANAR 3-SAT using the gadget in Figure 2(c).

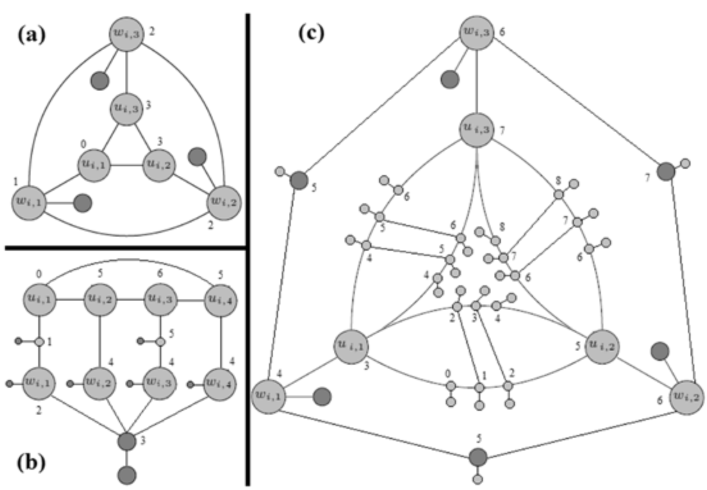

Figure 2.1. Gadget for each clause C_i.

Now we present our polynomial results. It is clear that $t(G) \geq 1$ if and only if G has a vertex of degree ≥ 2. The next result characterizes the graphs with $t(G) \geq 2$. The same question for $t(G) \geq 3$ is still open.

Theorem 2.2. *Let G be a graph. Then $t(G) \geq 2$ if and only if there exist $u \in V(G)$ and a neighbour v of u such that $A \cup \{v\}$ is a hull set, where $A \subset V(G)$ contains every vertex which is neither u nor a neighbour of u.*

Now we show how to determine $t(T)$ in linear time, for any tree T. Given two adjacent vertices $u, v \in V(T)$, denote by $s(u, v)$ the maximum time that u enters in the convex hull of S, among all hull sets S of the subtree of $T - v$ containing u; and by $t(u)$ the maximum time that u enters in the convex hull of S, among all hull sets S of T. Clearly, $t(T)$ equals $max_{u \in V(T)} t(u)$ and the values $s(u, v)$, $t(u)$ are given by:

$$s(u,v) = \begin{cases} 0, & \text{if } |N(u)| \leq 2, \\ 1 + \text{the second value in the non-decreasing} & \\ \text{ordering of the set } \{s(x,u) : x \in N(u) \setminus \{v\}\}, & \text{if } |N(u)| \geq 3. \end{cases}$$

$$t(u) = \begin{cases} 0, & \text{if } |N(u)| \leq 1, \\ 1 + \text{the second value in the non-decreasing} & \\ \text{ordering of the set } \{s(x,u) : x \in N(u)\}, & \text{if } |N(u)| \geq 2. \end{cases} \tag{2.1}$$

In order to compute the values $s(u, v)$ and $s(v, u)$, for each edge $uv \in E(T)$, we use a directed graph D with vertex set $V(T)$ and edges $\{(u, v), (v, u) \mid uv \in E(T)\}$. Observe that $s(u, v)$ can be computed only after all values $s(x, y)$ are known, where (x, y) is an arc of $D - v$ belonging to a directed path from some leaf of T to u. Thus, consider a partition of the arcs of D into sets $S_0, \ldots, S_{d-1}$, where d is the diameter of T, and S_i contains the arcs (u, v) such that the longest directed path of $D - v$ from some leaf to u has length i. In the beginning, we know $s(u, v)$, for all arc $(u, v) \in S_0$. Further, as long as $s(x, y)$ is known for each arc $(x, y) \in \bigcup_{j=0}^{i-1} S_j$, we can compute $s(u, v)$, for every arc $(u, v) \in S_i$, $i \in [1, d - 1]$. Therefore, we can compute all values $s(u, v)$ in linear time. With some modifications, we can adapt these ideas to obtain a polynomial time algorithm for chordal graphs.

Theorem 2.3. *If T is a tree, then $t(T)$ can be computed in linear time. Let G be a chordal graph. If G is 2-connected, then $t(G)$ can be computed in time $O(n^2 m)$; otherwise, $t(G)$ can be computed in time $O(n^2 m^2)$.*

Considering $(q, q - 4)$-graphs, q fixed, we proved that the percolation time $t(G)$ is bounded by q, and we give linear time algorithms to obtain $t(G)$. We mention that these graphs generalize the P_4-sparse graphs.

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