# The maximum infection time in the geodesic and monophonic convexities☆

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## Abstract

Recent papers investigated the maximum infection time $t_{P3}(G)$ of the $P_3$-convexity (also called maximum time of 2-neighbour bootstrap percolation) and the maximum infection time $t_{mo}(G)$ of the monophonic convexity. In 2014, it was proved that, for bipartite graphs, deciding whether $t_{P3}(G) \geq k$ is polynomial time solvable for $k \leq 4$, but is NP-complete for $k \geq 5$ [23]. In 2015, it was proved that deciding whether $t_{mo}(G) \geq 2$ is NP-Complete even for bipartite graphs [12]. In this paper, we investigate the maximum infection time $t_{gd}(G)$ of the geodesic convexity. We prove that deciding whether $t_{gd}(G) \geq k$ is polynomial time solvable for $k = 1$, but is NP-complete for $k \geq 2$ even for bipartite graphs. We also present an $O(n^3 m)$-time algorithm to determine $t_{gd}(G)$ and $t_{mo}(G)$ in distance-hereditary graphs. For this, we characterize all minimal hull sets of a general graph in the monophonic convexity. Moreover, we improve the complexity of the fastest known algorithm for finding a minimum hull set of a general graph in the monophonic convexity.

*Keywords:* Geodesic convexity, maximum infection time, monophonic hull number, monophonic convexity

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## 1. Introduction

A family $\mathcal{C}$ of subsets of a finite set $V$ is a *convexity on* $V$ if $\emptyset$ and $V$ belong to $\mathcal{C}$ and $\mathcal{C}$ is closed under intersections. A member of $\mathcal{C}$ is said to be a *convex set of* $\mathcal{C}$ or simply a $\mathcal{C}$*-convex set*. When there is no risk of confusion, we omit $\mathcal{C}$ and write only *convex set*. When $V$ is the vertex set of a graph $G$, a standard way to define a convexity $\mathcal{C}$ on $V$ is by fixing a family $\mathcal{P}$ of paths of $G$ and letting a subset $S$ of $V$ be $\mathcal{C}$-convex if and only if for every path $P$ in $\mathcal{P}$ whose endpoints belong to $S$ we have that all vertices of $P$ also belong to $S$. In that case we have a *graph convexity*. The most studied graph convexities are the *geodesic convexity* [17, 19], the *monophonic convexity* [14, 16, 18] and the $P_3$ *convexity* [5], where $\mathcal{P}$ is, respectively, the family of all shortest paths, of all induced paths, and of all paths of order three of the graph. A rich source on abstract convexities is a book by van de Vel [26].

The $\mathcal{C}$*-convex hull of* $S$ is the smallest $\mathcal{C}$-convex set, denoted $H_{\mathcal{C}}(S)$, containing $S$. We say that $S$ is a $\mathcal{C}$*-hull set* if $H_{\mathcal{C}}(S) = V(G)$. The $\mathcal{C}$*-hull number* is the cardinality of the minimum $\mathcal{C}$-hull set and is denoted by $h_{\mathcal{C}}(G)$. The $\mathcal{C}$*-interval of* $S$, denoted $I_{\mathcal{C}}(S)$, consists of $S$ and all vertices lying in some path of $\mathcal{P}$ that has endpoints in $S$. Observe that, since $G$ is finite, the $\mathcal{C}$-convex hull of $S$ can be obtained by starting with $S$ and repeatedly applying the $\mathcal{C}$-interval function until we obtain a $\mathcal{C}$-convex set.

We may also see the above process, used to obtain the $\mathcal{C}$-convex hull of $S$, as an infection that starts at the set $S$ and spreads to other vertices through the paths (of $\mathcal{P}$) that connect two infected vertices. Here, we are interested in the maximum amount of time needed to infect all vertices starting with a $\mathcal{C}$-hull set, where one unit of time corresponds to applying the interval function once. More precisely, let $I^0_{\mathcal{C}}(S) = S$ and $I^k_{\mathcal{C}}(S) = I_{\mathcal{C}}(I^{k-1}_{\mathcal{C}}(S))$ for $k \geq 1$. We say that a hull set $S$ takes time $k$ to infect $G$ if $I^k_{\mathcal{C}}(S) = V(G)$ but $I^{k-1}_{\mathcal{C}}(S) \neq V(G)$ (when $S = V(G)$ we say it takes time 0 to infect $G$). The *infection time of* $G$ *relative to* $\mathcal{C}$, called $t_{\mathcal{C}}(G)$, is the maximum $k$ such that there is a $\mathcal{C}$*-hull set* $S$ which takes time $k$ to infect $G$. We considered the decision version of this problem.

MAX Infection time on convexity $\mathcal{C}$
*Input:* A graph $G$ and an integer $k$.
*Question:* Is $t_{\mathcal{C}}(G) \geq k$?

There is vast literature about "infection problems", also studied under the names "dissemination", "diffusion" or "conversion" [2, 3, 21]. However, the way such infections spread varies considerably. For example, another common model is $r$-neighbour bootstrap percolation in which a vertex becomes infected if it has at least $r$ infected neighbors, a model introduced by Chalupa, Leath and Reich [10] that has found many applications in physics and computer science [15]. There are also applications in clustering phenomena, sandpiles [20], and many other areas of statistical physics, as well as in neural networks [1].

The question about the maximum infection time was originally posed by Bollobás for the 2-neighbour bootstrap percolation in the square grid, and solved by Benevides and Przykucki [8]. We remark that in general, counterintuitively, a hull set $S$ that infects $G$ in time $t_{\mathcal{C}}(G)$ may need to have more than $h_{\mathcal{C}}(G)$ elements. That is, a set that infect $G$ at the maximum possible time does not have to be of minimum order (although it has to be minimal). This follows from the main results in [7] and [8], even when $G$ is a grid and $\mathcal{C}$ is the $P_3$-convexity.

Note that the 2-neighbour bootstrap percolation model coincides with the infection problem for the $P_3$ convexity. The algorithm complexity of MAX INFECTION TIME ON THE $P_3$ CONVEXITY was considered in [6] where it was shown that it is NP-complete for general graphs and any fixed $k \geq 4$, and it was given polynomial algorithms for planar graphs, trees, chordal graphs, and for $k \leq 2$. In 2014, Marcilon et al. obtained polynomial time algorithms for $k \leq 3$ in general graphs and $k \leq 4$ in bipartite graphs, and proved that it is NP-complete for $k \geq 5$ in bipartite graphs [23]. In 2015, Marcilon and Sampaio proved that it is NP-complete even in grid graphs with maximum degree 3 and it is polynomial time solvable in solid grid graphs with maximum degree 3 [24]. The question about solid grid graphs with maximum degree 4 is still open.

Regarding the monophonic convexity, it was proved in 2015 that the maximum infection time problem is NP-complete for $k \geq 2$ in bipartite graphs [12]. In 2010, Dourado et al. obtained an $O(n^3m)$-time algorithm to compute a minimum hull set for the monophonic convexity [14].

In this work, we consider the MAX INFECTION TIME on the geodesic and on the monophonic convexities. This problem is, therefore, at the intersection of two large branches: "convexity problems" and "infection problems". In Section 3, we prove that the MAX INFECTION TIME ON GEODESIC CONVEXITY is NP-complete even if the input graph is bipartite and $k \geq 2$ is fixed.

In Section 4, we obtain an important characterization of all minimal hull sets in the monophonic convexity (see Theorem 4.9). We also consider the problem of determining the hull number of $G$ in the monophonic convexity. As a matter of fact, we observe that the clique decomposition presented in [22] can be used to improve the time complexity of the fastest known algorithm [14] for this problem from $O(n^3m)$-time to $O(nm)$-time. These results are useful for a polynomial time algorithm, which is given in Section 5, for computing the maximum infection time on distance-hereditary graphs, a class of graphs where the geodesic and monophonic convexities coincide. In the next section, we present some useful definitions and examples.

## 2. Preliminaries and notation

We consider only finite, simple, and undirected graphs. For a vertex $u$ of a graph $G$, denote by $N(u)$ the set of neighbors of $u$, and for $S \subseteq V(G)$, denote by $N(S)$ the set $\{v : v \in N(u) \setminus S \text{ and } u \in S\}$. The *subgraph of* $G$ *induced by* $S$ is denoted by $G[S]$. For two vertices $u, v \in V(G)$, the distance of $u$ to $v$, denoted $d(u, v)$, is the number of edges in a shortest path between $u$ and $v$. If every two vertices of $S$ are adjacent, then $S$ is called a *clique of* $G$; and if every two vertices are not adjacent, then $S$ is called an *independent set of* $G$. A graph is *bipartite* if its vertex set can be partitioned into two independent sets. A vertex $v$ is called simplicial if $N(v)$ is a clique. For any positive integer $n$, we define $[n]$ to be the set $\{t : t \text{ is integer and } 1 \leq t \leq n\}$.

When there is no ambiguity about which graph convexity we are using, we will drop the symbol of the convexity. For example, we will use $H(S)$ to denote the convex hull of $S$ in the considered underling convexity. On the other hand, sometimes only the type (e.g., geodetic, monophonic or $P_3$) of convexity is clear but not the convexity itself. For example, it may be clear that we are working on the geodesic convexity, but the family $\mathcal{C}$ of convex sets still depends on the underling graph $G$: a set $S$ may be convex in a graph $G$ but not in a subgraph $G'$ of $G$, even if $S \subseteq V(G')$, and vice-versa. Therefore, in some places, it will be useful to have a subscript to identify which graph was used to define the convexity.

As an example of the infection process, we present a family of (distance-hereditary) graphs $H_k$, for $k \geq 1$, where $t(H_k) = k$ in either the geodesic or monophonic convexity. This family will be useful in the proof that the Max infection time on geodesic convexity problem is $\mathsf{NP}$-complete for bipartite graphs (Theorem 3.1). The graph $H_k$ is defined recursively as

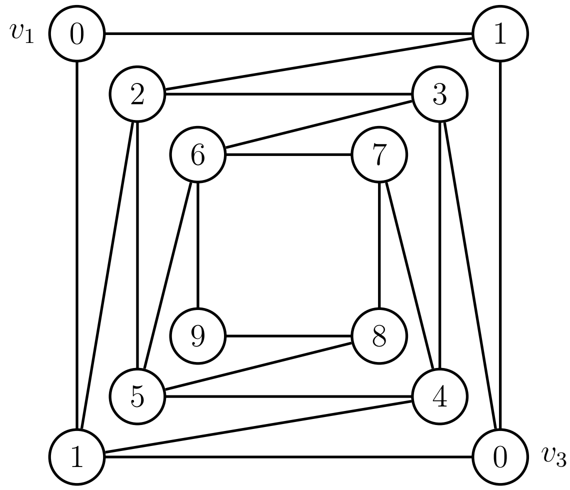


Figure 1: The Graph $H_9$, which is distance-hereditary. A value inside the vertex is the time which the vertex becomes infected starting from the hull set $v_1, v_3$.

follows (see also Figure 1): $H_1$ is a cycle of length four, where $V(H_1) = \{v_1, v_2, v_3, v_4\}$ and $E(H_1) = \{v_1v_2, v_2v_3, v_3v_4, v_4v_1\}$, and for $k \geq 2$ we set

$$V(H_k) = V(H_{k-1}) \cup \{v_{k+3}\} \text{ and} \tag{1}$$
$$E(H_k) = E(H_{k-1}) \cup \{v_{k+3}v_k, v_{k+3}v_{k+2}\}. \tag{2}$$

For our $\mathsf{NP}$-completeness result, we will make a polynomial reduction from the following well known $\mathsf{NP}$-complete problem [11].

3Sat
*Input:* A formula $\mathcal{F}$ in the CNF such that each clause has size 3.
*Question:* Is there a truth assignment satisfying all clauses of $\mathcal{F}$?

## 3. Geodetic infection time on Bipartite graphs

Throughout this section, we only consider the geodesic convexity. Our goal is to prove the following theorem.

**Theorem 3.1.** MAX Infection time on geodesic convexity *is* $\mathsf{NP}$-*complete even if the input graph is known to be bipartite and* $k \geq 2$ *is fixed.*

*Proof.* Let $G$ be a graph, $S$ be any subset of its vertices, and $k \geq 1$ be an integer. Since the set $I(S)$, result of the interval function applied to $S$, can be obtained in polynomial time on the size of $G$ [13], for a given $k$ one can check in polynomial time if $I^k(S) = V(G)$. Therefore this problem belongs to NP.

For the proof of the hardness, we first treat the case where $k = 2$. We shall do a reduction from 3SAT. Consider a boolean formula in the conjunctive normal form with $m$ clauses, say $\mathcal{F} = \{C_1, \ldots, C_m\}$, on the set of variables $\{x_1, \ldots, x_n\}$.

For each clause $C_i$ of $\mathcal{F}$ we build a clause gadget, the bipartite graph depicted in Figure 2, whose vertex set is $\{u_i, t_i\} \cup \{v_{i,l}, w_{i,l}, x_{i,l}, y_{i,l} : l \in [3]\}$, and edge set is $\{u_i t_i\} \cup \{u_i v_{i,l}, t_i w_{i,l}, v_{i,l} w_{i,l}, w_{i,l} x_{i,l}, x_{i,l} y_{i,l} : l \in [3]\}$.

We construct a graph $G$ as follows. Add each clause gadget to $G$ and further add vertices $q$, $r$ and $s$ along with the edges $rx_{i,l}$, $st_i$ for every $i \in [m]$ and $l \in [3]$. Finally, for every pair of literals $\ell_{i,a} \in C_i$ and $\ell_{j,b} \in C_j$, where $i \neq j$ and $a, b \in [3]$, such that $\ell_{i,a}$ and $\ell_{j,b}$ are not the negation of each other, add vertices $o_{i,a,j,b}$, $p_{i,a,j,b}$ along with the edges $o_{i,a,j,b}p_{i,a,j,b}$, $o_{i,a,j,b}q$, $o_{i,a,j,b}w_{i,a}$, $o_{i,a,j,b}w_{j,b}$, to complete the construction of $G$. See Figure 3 for a *partial* construction of the graph $G$, when $\mathcal{F} = \{C_1, C_2, C_3\}$ where $C_1 = \{x_1, x_2, x_4\}$, $C_2 = \{\overline{x_2}, x_3, \overline{x_4}\}, C_3 = \{\overline{x_1}, x_2, \overline{x_4}\}$. Note that the gray and white vertices define a bipartition of $G$.

We let $W = \{w_{i,l} : \text{ for } i \in [m], l \in [3]\}$, $T = \{t_i : \text{ for } i \in [m]\}$. Similarly $O$ is the set of vertices $o_{i,a,j,b}$ for all values of $i, a, j, b$ for which $o_{i,a,j,b}$ is defined. Define $X$, $Y$, $V$, $U$ and $P$ in a analogous way. Finally for every $i \in [m]$, let $D_i = \{u_i, v_{i,1}, v_{i,2}, v_{i,3}\}$.

Now we show that there exists a truth assignment $\mathcal{A}$, to the variables $\{x_1, \ldots, x_n\}$, satisfying all clauses of $\mathcal{F}$ if and only if the infection time of $G$ is at least 2. This shall be a consequence of a few claims.

**Claim 1.** *Every hull set of $G$ contains at least one vertex of $D_i$, for every $i \in [m]$.*

*Proof.* Fix $i \in [m]$. It suffices to show that $V(G) \setminus D_i$ is a convex set of $G$. Define $E_i = N(D_i) = \{t_i, w_{i,1}, w_{i,2}, w_{i,3}\}$. Observe that $V(G) \setminus D_i$ is a convex set if and only if $I(E_i) \cap D_i = \emptyset$. We distinguish two possibilities for a pair of vertices $z, z' \in E_i$. The first one is $z = t_i$ and $z' = w_{i,a}$, for some $a \in \{1, 2, 3\}$. In this case $zz' \in E(G)$ and then $I(z, z') = \{z, z'\}$. The second one is $z = w_{i,a}$ and $z' = w_{i,b}$, for some $a, b \in \{1, 2, 3\}$ with $a \neq b$. Since

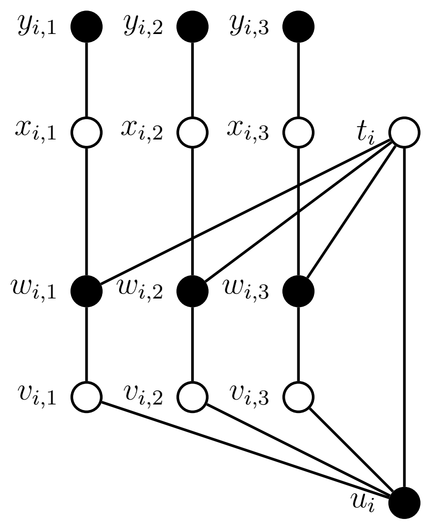


Figure 2: Clause gadget.

$t_i \in N(z) \cap N(z')$, it holds $d(z, z') = 2$. In fact, $t_i$ is the unique common neighbour of $z$ and $z'$. Then, $I(z, z') = \{z, z', t_i\}$. □

**Claim 2.** *Let $z \in D_i$, for some $i \in [m]$. Then $D_i \cup \{t_i\} \subset I(\{z, y_{i,1}, y_{i,2}, y_{i,3}\})$.*

*Proof.* If $z = u_i$, it is clear that $zv_{i,a}w_{i,a}x_{i,a}y_{i,a}$ is a shortest path between $z = u_i$ and $y_{i,a}$, for any $a \in \{1, 2, 3\}$,; and that $zt_iw_{i,1}x_{i,1}y_{i,1}$ is a shortest path between $z = u_i$ and $y_{i,1}$ (in $G$).

Now, if $z = v_{i,a}$, for some $a \in \{1, 2, 3\}$, it is also clear that $zu_iv_{i,b}w_{i,b}x_{i,b}y_{i,b}$, and $zu_it_iw_{i,b}x_{i,b}y_{i,b}$ are shortest paths between $z = v_{i,a}$ and $y_{i,b}$, for any $b \neq a$, with $b \in \{1, 2, 3\}$. □

**Claim 3.** *$W \cup X \cup O \cup \{q, r\} \subset I(Y \cup P)$ and $s \notin I(Y \cup P)$.*

*Proof.* For every vertex $z \in W \cup X \cup O \cup \{q, r\}$ we show that there exists a pair of vertices $z', z'' \in Y \cup P$ such that $z \in I(\{z', z''\})$. Recall that that $G$ is bipartite, and color its vertices with "black" or "white" as in Figure 3. Note that $Y \cup P$ is a subset of the "black" vertices. Also, for two "black" vertices $z'$, $z''$, we have that $d(z', z'') < 4$ if and only if $z'$ and $z''$ have a common neighbor.

Therefore, for any two distinct vertices $z', z'' \in Y$, we have $d(z', z'') = 4$. Since there is a path of length 4 between $z'$ and $z''$ that pass through $r$, we

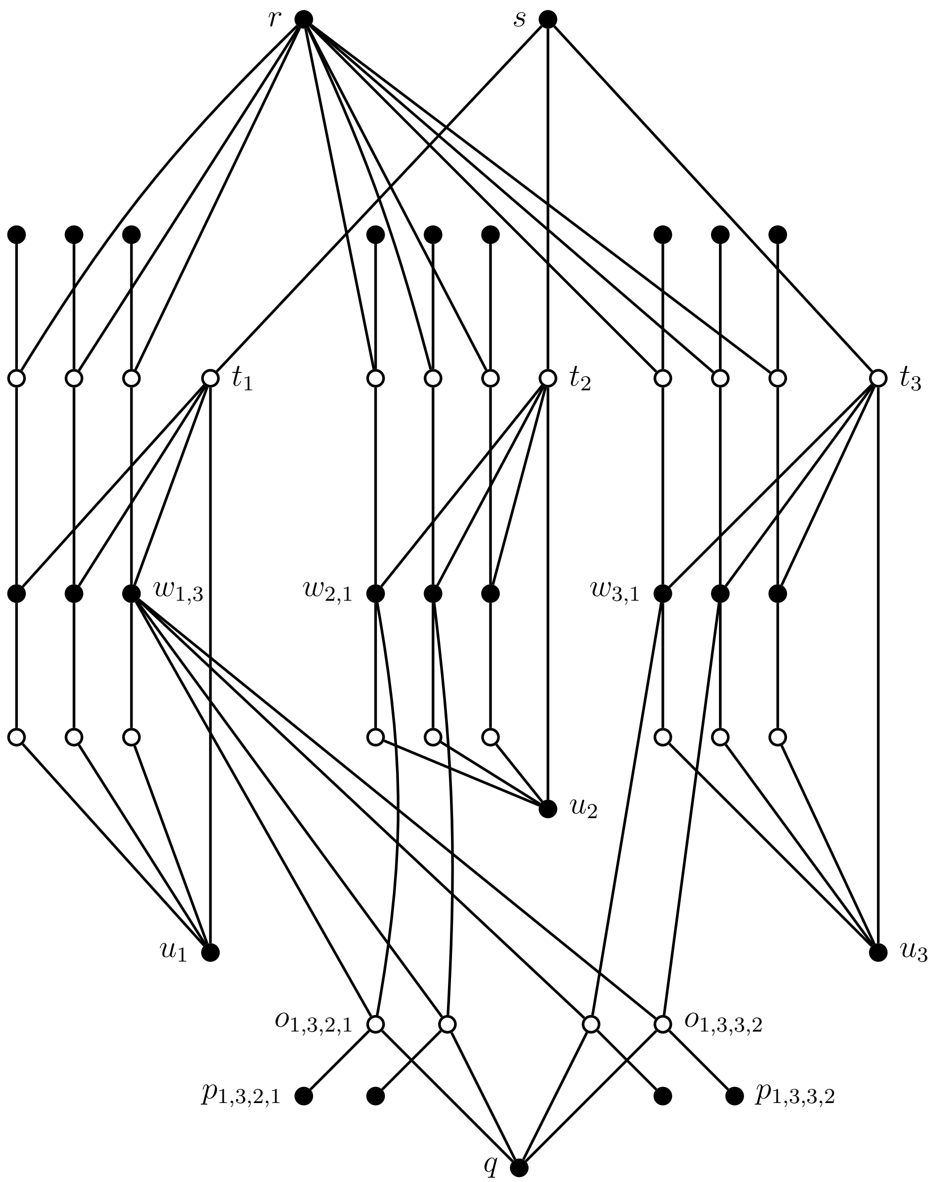


Figure 3: Partial construction of $G$ for $\mathcal{F} = \{C_1, C_2, C_3\}$ where $C_1 = \{x_1, x_2, x_4\}$, $C_2 = \{\overline{x_2}, x_3, \overline{x_4}\}, C_3 = \{\overline{x_1}, x_2, \overline{x_4}\}$. The figure does not contain all vertices of $O$ and $P$. In fact, it shows only vertices of $O$ and $P$ related with the literal $x_4$ of $C_1$.

have that $r \in I(\{z', z''\})$. Similarly, for any two distinct vertices $z', z'' \in P$, we have that $q \in I(\{z', z''\})$.

Let $i \in [m]$ and $a \in \{1, 2, 3\}$. Since there exists at least one pair $(j, b)$ for which $p_{i,a,j,b}$ is a vertex, there exists the shortest path $y_{i,a}x_{i,a}w_{i,a}o_{i,a,j,b}p_{i,a,j,b}$. Then one can easily check that $W \cup X \cup O \subset I(Y \cup P)$.

It remains to show that $s \notin I(Y \cup P)$. Note that for any $p \in P$ and $y \in Y$, we have that $d(q, p) = 2$, $d(r, y) = 2$ and $d(q, y) = 4$. This implies that the distance between any two vertices of $Y \cup P$ is at most 6. Since $d(z', s) = 4$ for any vertex $z' \in Y \cup P$, we have that any path between two vertices of $Y \cup P$ that pass through $s$ has length at least 8. Therefore $s \notin I(Y \cup P)$ and the claim holds. □

**Claim 4.** *$V(G) \setminus \{s\} \subseteq I(S)$, for every hull set $S$ of $G$.*

*Proof.* Since all vertices of $Y$ and $P$ are pendant vertices we have $Y \cup P \subset S$ for any hull set $S$. Then, by Claim 3, it remains to show only that $V \cup U \cup T \subset I(S)$. By Claim 1, there exists $v \in S \cap D_i$, for every $i \in [m]$. Applying Claim 2 for each such $v$, we complete the proof of Claim 4. □

**Claim 5.** *Let $v_{i,a}$ and $v_{j,b}$ with $i \neq j$. Then $s \in I(\{v_{i,a}, v_{j,b}\})$ if and only if $\ell_{i,a}$ is the negation of $\ell_{j,b}$.*

*Proof.* Note that $d(v_{i,a}, s) = d(v_{j,b}, s) = 3$. Then, $s \notin I(\{v_{i,a}, v_{j,b}\})$ if and only if $d(v_{i,a}, v_{j,b}) < 6$, i.e., $d(v_{i,a}, v_{j,b}) = 2$ or $d(v_{i,a}, v_{j,b}) = 4$. The first case is not possible because there is no possibility for $v_{i,a}$ and $v_{j,b}$ share a neighbour. The other possibility occurs if and only if two vertices, one neighbour of each one, share a neighbour. Since $N(v_{i,a}) = \{w_{i,a}, u_i\}$ and $N(v_{j,b}) = \{w_{j,b}, u_j\}$, the path of length 4 must be $v_{i,a}w_{i,a}zw_{j,b}v_{j,b}$. By the construction, $z = o_{i,a,j,b}$ and $\ell_{i,a}$ is the negation of $\ell_{j,b}$. □

Now, let us finish the proof of Theorem 3.1. Let $S$ be a hull set of $G$ such that $I(S)$ is a proper subset of $V(G)$, that is, $t(S) > 1$. By Claim 4, $I(S) = V(G) \setminus \{s\}$. By Claim 1, for every $i \in [m]$, the set $S$ contains some vertex of $D_i$. Suppose, by contradiction, that for some $j \in [m]$, it holds $u_j \in S \cap D_j$. Observe that $s$ is in a shortest path between $u_j$ and $u$, where $u$ is some vertex of $S \cap D_k$, for any $k \neq j$, which is a contradiction. Then, for every $i \in [m]$, $S$ contains $v_{i,a_i}$, where $a_i \in \{1, 2, 3\}$. By Claim 5, every pair of literals $\ell_{i,a} \in C_i$ and $\ell_{j,b} \in C_j$ associated to vertices $v_{i,a}$ and $v_{j,b}$ contained in $S$ are not the negation of each other. Then, the truth assignment where $\ell_{i,a}$ is true if and only if $v_{i,a}$ belongs to $S$ satisfies all clauses of $\mathcal{F}$.

Conversely, let $\mathcal{A}$ be a truth assignment for $\{x_1, \ldots, x_n\}$ satisfying all clauses of $\mathcal{F}$. Consider the set $S$ formed by the pendant vertices of $G$ plus one vertex $v_{i,a}$ for every clause $C_i$ of $\mathcal{F}$, where $v_{i,a}$ is chosen if $\ell_{i,a}$ assumes value true by $\mathcal{A}$. By Claim 3 and Claim 2, $V(G) \setminus \{s\} \subseteq I(S)$. And, since $s$ is not a simplicial vertex of $G$, it follows that $S$ is a hull set of $G$. By Claim 5, it remains to show that $s \notin I(u, v)$ for $u \in Y \cup P$ and $v \in V$. Note that $d(u, s) = 4$ and $d(v, s) = 3$, but $d(u, v) \leq 5$, completing the proof for $k = 2$.

Now suppose that $k > 2$. To finish the proof, it suffices to add to the construction of $G$ a vertex $s'$ twin to $s$ and a copy of the graph $H_{k-2}$ (as defined by equations (1) and (2) at the end of Section 2), such that $s'$ and $s$ are identified to the vertices $v_1$ and $v_3$ of the copy of $H_{k-2}$, respectively. It is not hard to check that this new graph $G$ has infection time $k$ if and only if $\mathcal{F}$ is satisfiable. $\square$

## 4. Minimal and minimum hull sets in the monophonic convexity

Here, we call any hull set in the monophonic convexity by *m-hull set*. In this section we show how to use results of Leimer [22], about the clique separator decomposition, to improve the complexity of the fastest known (polynomial) algorithm [14] for finding a minimum $m$-hull set of a general graph. We also present a characterization of all minimum and of all minimal $m$-hull sets of general graphs. The later result is essential to our polynomial time algorithm for finding the maximum infection time in a distance-hereditary graph, that will be presented in Section 5.

A survey on clique separator decompositions can be found in [9]. Let us start with some preliminary notations. Given a graph $G$ and a set $C \subseteq V(G)$, we say that $C$ is a *separator of* $G$ if there are non-empty sets $A, B \subset V(G) \setminus C$ such that every path in $G$, between some $a \in A$ and $b \in B$, contains a vertex in $C$. Separators that are cliques are called *clique separators*. If $C$ is a clique separator of $G$ and $H$ is any connected component of $G - C$, the subgraph $G[V(H) \cup C]$ is called an *$C$-component of* $G$. The family of $C$-components of $G$ is a *decomposition of* $G$. We say that $G$ is *reducible* if it contains a clique separator, otherwise it is called a *prime*. We note that, in the literature, those are also called atoms. A *maximal prime subgraph of* $G$, or simply *mp-subgraph of* $G$, is a maximal induced subgraph of $G$ that is prime.

A separator $C$ is a *minimal separator of* $G$ if no proper subset of $C$ is a separator of $G$. And $C$ is a *relative minimal separator for* $G$ if there exist vertices $v, w \in V(G)$ such that $v$ and $w$ are in different components of $G - C$, but for every proper subset $C'$ of $C$ we have that $u$ and $w$ are in the same component of $G-C'$. Note that every minimal separator is a relative minimal separator for $G$, but the reverse is not true.

Assume that a reducible graph $G$ is decomposed by a clique $C$ and each $C$-component is decomposed further until all derived subgraphs are prime. A family of prime subgraphs of $G$ that can be derived in that way will be called a *derived family.*

**Lemma 4.1.** *Let $C$ be a clique of $G$ and $G_1$ be a $C$-component. Then any separator of $G_1$ is also a separator of $G$. Furthermore, if $S$ is a relative minimal separator of $G_1$ then it is a relative minimal separator of $G$.*

*Proof.* Let $G$, $C$, and $G_1$ be defined as in the statement. Let $S$ be a separator of $G_1$. So there exist vertices $x, y \in G_1$ such that any path from $x$ to $y$ in $G_1$ pass through $S$. Suppose that there is a path in $G$ from $x$ to $y$ that avoids $S$. Since $C$ is a clique, we can shorten this path to a path in $G_1$ (that shall also avoids $S$). Therefore, $S$ also separates $x$ from $y$ in $G$. For the last part, it is trivial that if $S$ is minimal for $x, y$ in $G_1$ then it is also minimal for $x, y$ in $G$. □

By the previous proposition, we also notice that if $G_1$ is a $C$-component of $G$, where $C$ is a clique, and $x, y \in V(G_1)$ then any induced path from $x$ to $y$ in $G$ must have all its vertices in $G_1$. That is the intuition on why the clique separator decomposition is usefull to solve problems in the monophonic convexity. The following result of [14] indicates why this decomposition is fruitful for finding $m$-hull sets.

**Theorem 4.2.** [14] *If $G$ is a prime graph that is not a complete graph, then every pair of non-adjacent vertices is an $m$-hull set of $G$.*

Consider a graph $G$ and a clique separator of $C$. Clearly, any *mp*-subgraph of a graph is "unbreakable" under a $C$-decomposition (that is, it must be contained in a $C$-component). Therefore, every *mp*-subgraph of $G$ is an *mp*-subgraph of some $C$-component of $G$. The converse is not always true. But we have the following result.

**Lemma 4.3.** *A derived family is exactly the family of mp-subgraphs of $G$ if and only if it is obtained by a decomposition that uses only cliques that are relative minimal separators of $G$.*

*Proof.* This follows directly from (2.2) of [22] together with itens (i) and (iii) of Theorem 4.1 of [22]). $\square$

In our proofs, we will also need the following concepts. Given any graph $G$, consider an (arbitrary) ordering of the *mp*-subgraphs $F_1, \ldots, F_t$ of $G$ and then define $R_i = V(F_i) \cap (V(F_1) \cup \ldots \cup V(F_{i-1}))$, for $i \in [t]$. This ordering of the *mp*-subgraphs is a *D-ordering* if, for all $i \in \{2, \ldots, t\}$, there is $j$ such that $j < i$ and $R_i \subseteq V(F_j)$. According to Theorem 2.5 of [22], there is a $D$-ordering for the *mp*-subgraphs of any graph. Further, Proposition 2.4 of [22] says that every permutation of the *mp*-subgraphs of $G$ that is a $D$-ordering has the same family of $R_i$ sets. Then, one can define $\mathcal{R}(G) = \{R_2, \ldots, R_t\}$ and $R(G) = R_2 \cup \ldots \cup R_t$, taking as starting point any $D$-ordering of the *mp*-subgraphs of $G$. We will also use the following result of Leimer (again see Theorem 4.1 of [22], itens (i) and (iv)).

**Theorem 4.4.** [22] *Let $C$ be a set of vertices of a graph $G$. We have that $C$ is a clique and a relative minimal separator for $G$ if and only if $C \in \mathcal{R}(G)$.*

In our algorithm, we shall adapt results of [14]. There, the authors have used a decomposition of the following type. Take a minimal clique $C$ of $G$, look at the $C$-components of $G$, say $G_1, \ldots, G_t$. For each component take a minimal clique separator of that component and continue the process recursively to obtain a derived family. Note that the minimal separator of the components does not have to be a minimal separator of $G$. After that, they classify each set in the derived family into certain types and used this classification to find an $m$-hull set of $G$. We observe that any minimal clique separator in $G_1$ is a relative minimal clique for $G_1$. By Lemma 4.1, it is also a relative minimal separator of $G$. By the same argument, for any component obtained at further steps in that decomposition, any minimal clique separator of that component is a relative minimal clique separator of $G$. By Lemma 4.3, the derived family obtained in [14] is precisely the family of *mp*-subgraphs.

As a consequence of the above, we can rewrite the characterization of minimum $m$-hull sets presented in [14]. Let $F$ be any *mp*-subgraph. We say that $F$ is:

**type** 0 if $V(F) \cap R(G)$ is not a clique;

**type** 1 if $V(F) \cap R(G)$ is a clique and there is a vertex $u \in V(F)$ not adjacent to some vertex of $V(F) \cap R(G)$;

**type** 2 if $V(F) \cap R(G)$ is a clique, $V(F)$ is not a clique, and every vertex of $V(F) \cap R(G)$ is universal in $F$;

**type** 3 if $V(F)$ is a clique.

As observed in [14], one can check that $F$ is of exactly one of the above types. This result implies that we can also rewrite the characterization of minimum $m$-hull sets from [14] using *mp*-subgraphs.

**Theorem 4.5.** *A set $S$ is a minimum $m$-hull set of a graph $G$ if and only if for every mp-subgraph $F$ of $G$, the set $S$ satisfies the following conditions:*

- *if $F$ is of type 0, then $S \cap V(F) = \emptyset$;*
- *if $F$ is of type 1, then $S \cap (V(F) \setminus R(G)) = \{u\}$ for a vertex $u$ not adjacent to some vertex of $V(F) \cap R(G)$);*
- *if $F$ is of type 2, then $S \cap (V(F) \setminus R(G)) = \{u, v\}$ for some pair $u, v$ of non-adjacent vertices;*
- *if $F$ is of type 3, then $V(F) \setminus R(G) \subseteq S$.*

The algorithm used in [14] to find the monophonic hull number of a graph on $n$ vertices and $m$ edges, had running time $O(n^3m)$ and consisted in finding minimal clique separators (using an algorithm for finding a clique separator of a graph [27]) and recursively obtaining the derived family. Now, using Theorem 4.5, the complexity analysis is the following. The *mp*-subgraphs and $\mathcal{R}(G)$ can be found in $O(nm)$ time using the algorithm of Leimer [22]. The number of *mp*-subgraphs of $G$ is $O(n)$ [22, 25]. The type of each *mp*-subgraph can be found in $O(m)$ time. To compute the hull number we only have to add 1 for each *mp*-subgraph of type 1, two for each of type 2 and $V(F)$, and $|V(F) \setminus R(G)|$ for each of type 3. We note that since the sets $V(F) \setminus R(G)$ are disjoint, there is no double counting in the previous summation. Therefore, one can find a minimum hull set of any graph in the monophonic convexity in $O(nm)$ steps.

In the sequel, we present a characterization of minimal $m$-hull sets. First, we state two simple lemmas from [14] rewriting them using the equivalence of the atoms in their decomposition with the *mp*-subgraphs.

**Lemma 4.6.** [14] *Let $F$ be an mp-subgraph of $G$. If $F$ is of type $t$, $t \in \{1, 2\}$, then every $m$-hull set of $G$ contains at least $t$ vertices of $V(F) \setminus R(G)$.*

**Lemma 4.7.** [14] *A vertex $v$ is simplicial in $G$ if and only if it belongs to a mp-subgraph of $G$ of type 3 and $v \notin R(G)$.*

Next, it is useful to show that $R(G)$ is contained in the interval of any $m$-hull set of $G$.

**Theorem 4.8.** *Let $G$ be a graph and $S$ a minimal $m$-hull set of $G$. Then $R(G) \cap S = \emptyset$ and $R(G) \subset I(S)$.*

*Proof.* For every $R_i \in \mathcal{R}(G)$, Theorem 4.4 says that $R_i$ is a minimal clique separator of $G$ relative to some pair of vertices $v_{i,1}, v_{i,2}$. Denote by $H_{i,1}$ and $H_{i,2}$ the $R_i$-components of $G$ containing $v_{i,1}$ and $v_{i,2}$, respectively. Note that the complement, in $G$, of $V(H_{i,j}) \setminus R_i$ is a $m$-convex set for $j \in 1, 2$. This implies that $S \cap (V(H_{i,j}) \setminus R_i)$ is non-empty. And, by the minimality of $R_i$, every vertex of $R_i$ has a neighbor in $H_{i,1}$ and in $H_{i,2}$. This implies that there are vertices $v'_{i,j} \in S \cap (V(H_{i,j}) \setminus R_i)$, for $j \in \{1, 2\}$ such that, for every $u \in R_i$ there is an induced path $P_1$ from $v'_{i,1}$ to $u$ in $H_{i,1}$ and an induced path $P_2$ from $v'_{i,2}$ to $u$ in $H_{i,2}$. Since $V(P_j) \cap R_i = \{u\}$, for $j \in \{1, 2\}$, it is clear that the concatenation of $P_1$ and $P_2$ is an induced path $P_u$ of $G$, concluding the proof that $R(G) \subset I(S)$. Since $S$ is minimal, we also have that $R(G) \cap S = \emptyset$. □

Now, we can present the characterization of minimal $m$-hull sets of general graphs.

**Theorem 4.9.** *A set $S$ is a minimal $m$-hull set of a graph $G$ if and only if for every mp-subgraph $F$ of $G$, $S$ satisfies the following conditions:*

- *if $F$ is of type 0, then $S \cap V(F) = \emptyset$;*
- *if $F$ is of type 1, then either $S \cap (V(F) \setminus R(G)) = \{u\}$ for a vertex $u$ not adjacent to some vertex of $V(F) \setminus R(G))$ or $S \cap (V(F) \setminus R(G))) = \{u, v\}$ for some pair $u, v$ of non-adjacent vertices;*
- *if $F$ is of type 2, then $S \cap (V(F) \setminus R(G)) = \{u, v\}$ for some pair $u, v$ of non-adjacent vertices;*
- *if $F$ is of type 3, then $V(F) \setminus R(G) \subseteq S$.*

*Proof.* Let $k$ be the number of $mp$-subgraphs of $G$, where $k \geq 1$. Let $S$ be a minimal $m$-hull set of $G$ and $F$ an $mp$-subgraph $F$ of $G$ of type $t$. By

Theorem 4.8, we know that $R(G) \cap S = \emptyset$, then the vertices of $F$ contained in $S$ belong to $V(F) \setminus R(G)$.

If $t = 0$, there are members $R_x$ and $R_y$ of $\mathcal{R}(G)$ contained in $F$ such that $R_x \cup R_y$ is not a clique. Defining the vertices $v_{x,1}, v_{x,2}, v_{y,1}, v_{y,2}$ as in the proof of Theorem 4.8, we can conclude that at least one of $v_{x,1}, v_{x,2}$ and one of $v_{y,1}, v_{y,2}$ does not belong to $V(F)$. Therefore, for every vertex $u$ of $R_x \cup R_y$ there is an induced path $P_u$ with endpoints in $S$ that pass through $u$, similar to the one constructed in the proof of Theorem 4.8. Then, since $S$ is minimal, $S$ does not contain any vertex of $F$.

If $t = 1$, by Lemma 4.6, the set $S$ contains at least one and at most two vertices of $F$. If $S \cap F$ contains one vertex, it is clear that it must be not adjacent to some vertex of $V(F) \cap R(G)$. Now, suppose that $S \cap F$ contains two vertices, say $x$ and $y$. If these two vertices were adjacent, then at least one of them, say $x$, must be not adjacent to some vertex $x'$ of some member $R_x \in \mathcal{R}(G)$ with $R_x \subseteq V(F) \cap R(G)$. Now, considering vertices $v'_{x,1}$ and $v'_{x,2}$ analogously defined as in the previous case, we can say that at least one of them, say $v_{x,1}$, is not contained in $V(F)$. Then, we can conclude that there is an induced path from $x$ to $v_{x,1}$ containing $x'$. This would imply that $y \in H(S \setminus \{y\})$. Therefore $xy \notin E(G)$.

If $t = 2$, by Lemma 4.6, the set $S$ contains two non-adjacent vertices of $V(F) \setminus R(G)$. Since these two vertices form an $m$-hull set of $G$ and $S$ is minimal, we have $|S \cap (F \setminus R(G))| = 2$.

The result for $t = 3$ follows directly from Lemma 4.7 and the fact that Theorem 4.8 implies that $S \cap R(G) = \emptyset$.

Conversely, let $S$ be a set of vertices of $G$ satisfying, for every *mp*-subgraph $F$ of $G$, the property associated with the type of $F$. If, for some *mp*-subgraph $F$ of $G$ of type 1, the set $S$ contains two non-adjacent vertices, then $V(F) \subseteq H(S)$, by Theorem 4.2. This implies that $H(S)$ contains a set satisfying all four conditions of Theorem 4.5, which implies that $S$ is an $m$-hull set of $G$. $\square$

## 5. Infection time on distance-hereditary graphs

A connected graph $G$ is called *distance-hereditary* if for every induced subgraph $G'$ of $G$ and every pair of vertices $u, v \in V(G')$ it holds that $d_G(u,v) = d_{G'}(u,v)$. This graph class admits a characterization by forbidden induced subgraphs and recognition in polynomial time [4]. Observe that geodesic and monophonic convexities coincide in distance-hereditary graphs, since for a graph in this class every induced path is also a shortest path. Although even the computation of the interval function is a NP-hard problem in the monophonic convexity for general graphs [14], the previous fact allows us to use the known algorithms for computing the interval function in the geodesic convexity together with the characterization of minimal $m$-hull sets for solving the maximum infection problem in distance-hereditary graphs. Next, we present a polynomial algorithm for that.

Since every hull set of a graph $G$ contains a minimal hull set, $t(G)$ can be determined by finding the maximum $t(S)$ among all minimal hull sets of $G$. Theorem 4.9 characterizes the minimal $m$-hull sets of $G$ according to their intersections with $mp$-subgraphs. The overall idea of our algorithm consists in decomposing $G$ into $mp$-subgraphs, and reducing the problem of computing the maximum infection time of $G$ to the problem of computing the infection time of a polynomial number of sets of these prime graphs.

We need of some definitions. Let $G$ be a graph, $S$ a hull set of $G$, and $F$ an $mp$-subgraph of $G$. Let

- $F^*$ be the graph obtained from $F$ by adding, for every $C_j \in \mathcal{R}(G)$ such that $C_j \subset V(F)$, a vertex $x_j$ and the edges $\{vx_j : v \in C_j\}$; and
- $S_F = (S \cap V(F)) \cup \{x_j : C_j \subseteq R(G) \cap V(F)\}$;

The following result shows that, for every $mp$-subgraph $F$ of a graph $G$ and any hull set $S$ of $G$, if we start an infection with $S$, we can determine which vertices of $F$ become infected at any given time by looking only at infected vertices in $S_F$.

**Lemma 5.1.** *Let $F$ be an mp-subgraph and $S$ an m-hull set of a graph $G$. Then $I^k_G(S) \cap V(F) = I^k_{F^*}(S_F) \cap V(F)$, for any $k \geq 0$.*

*Proof.* For $k = 0$, the definitions imply that $S \cap V(F) = S_F \cap V(F)$. For $k > 0$, observe that it suffices to show $I_G(S) \cap V(F) = I_{F^*}(S_F) \cap V(F)$. And this is equivalent to show that a vertex $v \in V(F) \setminus S$ is in an induced path of $G$ between two vertices of $S$ if and only if $v$ is in an induced path of $F^*$ between two vertices of $S_F$.

Let $w \in V(F) \setminus S$, and $P_{uv}$ be an induced $(u, v)$-path of $G$ passing through $w$ such that $u, v \in S$. Denote by $P_{uw}$ the subpath of $P_{uv}$ beginning at $u$ and ending at $v$. If $u \in V(F)$, define $P'_{uw} = P_{uw}$. Otherwise $P_{uw}$ contains a vertex $u'$ of some clique $C_i \in \mathcal{R}(G)$ such that $C_i \subset V(F)$. In this case, define $P'_{uw}$ as $x_i$ concatenated with the subpath of $P_{uw}$ beginning at $u'$ and ending at

$w$. Define $P_{vw}$ and $P'_{vw}$ analogously. It is clear that $P'_{uw}$ concatenated with $P'_{vw}$ is a induced path of $F^*$ passing through $w$ such that the extremities are two vertices of $S_F$.

Conversely, let $w \in V(F^*) \setminus S_F$, and $P_{uv}$ be an induced $(u, v)$-path of $F^*$ passing through $w$ such that $u, v \in S_F$. Denote by $P_{uw}$ the subpath of $P_{uv}$ beginning at $u$ and ending at $v$. If $u \in V(F)$, define $P'_{uw} = P_{uw}$. Otherwise $u = x_j$ for some $R_i \in \mathcal{R}(G)$ such that $R_i \subset V(F)$. Theorem 4.4 says that $R_i$ is a minimal separator of $G$ relative to some pair of vertices $v_{i,1}, v_{i,2}$. Without loss of generality we can say that the $R_i$-component $H_{i,1}$ containing $v_{i,1}$ does not contain $F$. Observe that $H_{i,1}$ is a proper $m$-convex sets of $G$ and that every vertex of $R_i$ has a neighbor in $H_{i,1}$. This implies that there is a vertex $v'_{i,1} \in S \cap (V(H_{i,1}) \setminus R_i)$. Then, consider an induced path $P_{i,1}$ of $H_{i,1} - R_i$ from $v'_{i,1}$ to some vertex $u''$ that is neighbor to some vertex $u' \in R_i \cap V(P_{uw})$. Now, define $P'_{uw}$ as the concatenation of the three paths $P_{i,1}$, the edge $u'u''$, and the subpath of $P_{uw}$ from $u'$ to $w$.

Next, define $P_{vw}$ and $P'_{vw}$ analogously. It is clear that $P'_{uw}$ concatenated with $P'_{vw}$ is a induced path of $G$ passing through $w$ such that the extremities are two vertices of $S$. $\square$

This result has the following important consequence for the algorithm.

**Corollary 5.2.** *Let $G$ be a graph. Then $t(G) = \max\{t_{F^*}(S_F) : S$ is a minimal $m$-hull set and $F$ is an mp-subgraph of $G\}$.*

We can now describe the algorithm for computing $t(G)$ of a distance-hereditary graph $G$ with $n$ vertices and $m$ edges in the geodesic or monophonic convexity. The *mp*-subgraphs and $\mathcal{R}(G)$ can be found in $O(nm)$ time [22]. The number of *mp*-subgraphs of $G$ is $O(n)$ [22, 25]. The type of each *mp*-subgraph can be determined in $O(m)$ time. By Corollary 5.2, for every *mp*-subgraph $F$ of $G$, we need to find the set maximizing $t_{F^*}(S_F)$ among all minimal hull sets $S$ of $G$. However, by Lemma 5.1, it suffices to consider all distinct intersections of $V(F)$ with all minimal hull sets of $G$. By Theorem 4.9, the number of such intersections is $O(n^2)$ for *mp*-subgraphs of types 1 and 2, and is $O(1)$ for *mp*-subgraphs of types 0 and 3. Finding $k$ such that $I^k(S) = H(S)$ can be done in $O(nm)$ steps for general graphs in the geodesic convexity [13] for any set $S$. Then, a direct analysis leads to an algorithm with time complexity $O(n^4m)$. However, we observe that the sum of the distinct intersections of all *mp*-subgraphs of $G$ with all minimal hull sets of $G$ is $O(n^2)$. Then, doing an amortized analysis, we conclude that the total complexity of this algorithm is $O(n^3m)$.

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