Edge-colorings of graphs avoiding complete graphs with a prescribed coloring

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Abstract

Given a graph F and an integer $r \geq 2$, a partition $\widehat{F}$ of the edge set of F into at most r classes, and a graph G, define $c_{r,\widehat{F}}(G)$ as the number of r-colorings of the edges of G that do not contain a copy of F such that the edge partition induced by the coloring is isomorphic to the one of F. We think of $\widehat{F}$ as the pattern of coloring that should be avoided. The main question is, for a large enough n, to find the (extremal) graph G on n vertices which maximizes $c_{r,\widehat{F}}(G)$. This problem generalizes a question of Erdős and Rothschild, who originally asked about the number of colorings not containing a monochromatic clique (which is equivalent to the case where F is a clique and the partition $\widehat{F}$ contains a single class). We use Hölder's Inequality together with Zykov's Symmetrization to prove that, for any $r \geq 2$, $k \geq 3$ and any pattern $\widehat{K_k}$ of the clique K_k, there exists a complete multipartite graph that is extremal. Furthermore, if the pattern $\widehat{K_k}$ has at least two classes, with the possible exception of two very small patterns (on three or four vertices), every extremal graph must be a complete multipartite graph. In the case that $r = 3$ and $\widehat{F}$ is a rainbow triangle (that is, where $F = K_3$ and each part is a singleton), we show that an extremal graph must be an almost complete graph. Still for $r = 3$, we extend a result about monochromatic patterns of Alon, Balogh, Keevash and Sudakov to some patterns that use two of the three colors, finding the exact extremal graph. For the later two results, we use the Regularity and Stability Method.

Keywords: Edge-coloring, Extremal Graph, Symmetrization, Hölder, Regularity

1. Introduction

For any fixed graph F, we say that a graph G is *F-free* if it does not contain F as a subgraph. Finding the maximum number of edges among all F-free n-vertex graphs, and determining the class of n-vertex graphs that achieve this number is known as the *Turán problem* associated with F, which was solved for complete graphs in [21]. The maximum number of edges in an F-free n-vertex graph is denoted by $\mathrm{ex}(n, F)$ and the n-vertex graphs that achieve this bound are called *F-extremal*. Turán has found the value of $\mathrm{ex}(n, F)$ for the case where F is a clique K_k on k vertices, for any $k \geq 3$. Moreover, he showed that the K_k-free

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graph on n vertices which has $\mathrm{ex}(n, K_k)$ edges is unique (up to isomorphism). This graph is a complete multipartite graph with $k-1$ parts of sizes as equal as possible, and we will denote it by $T_{k-1}(n)$. This problem and its many variants have been widely studied and there is a vast literature related with it. For more information, see Füredi and Simonovits [5] and the references therein.

In connection with a question of Erdős and Rothschild [4], several authors have investigated the following related problem. Instead of looking for F-free n-vertex graphs, they were interested in *edge-colorings* of graphs on n vertices such that *every color class is F-free.* (We observe that edge colorings in this work are not necessarily proper colorings.) More precisely, given an integer $r \geq 1$ and a graph F containing at least one edge, one considers the function $c_{r,F}(G)$ that associates, with the graph G, the number of r-colorings of the edge set of G for which there is no monochromatic copy of F. Similarly as before, the problem consists of finding $c_{r,F}(n)$, the maximum of $c_{r,F}(G)$ over all n-vertex graphs G.

The function $c_{r,F}(n)$ has been studied for several classes of graphs, such as complete graphs [1, 17, 23], odd cycles [1], matchings [7], paths and stars [9]. The hypergraph analogue of this problem has also been considered, see for instance [8, 10, 14, 15], and there has been recent progress in the context of additive combinatorics [6]. There is a straightforward connection between $c_{r,F}(n)$ and $\mathrm{ex}(n, F)$, namely

$$c_{r,F}(n) \geq r^{\mathrm{ex}(n,F)} \text{ for every } n \geq 2, \tag{1}$$

as any r-coloring of the edges of an F-extremal n-vertex graph is trivially F-free, and there are precisely $r^{\mathrm{ex}(n,F)}$ such colorings. For $r \in \{2, 3\}$ the inequality (1) is actually an equation for several graph classes, such as complete graphs [1, 23], odd cycles [1] and matchings [7]. On the other hand, for $r \geq 4$ and all connected F, one may easily show that $c_{r,F}(n) > r^{\mathrm{ex}(n,F)}$ (see [1] for non-bipartite graphs and [9, Proposition 3.4] for bipartite graphs).

Here we consider a natural generalization of the above, which was first studied by Lefmann and one of the current authors [11]. Given a k-vertex graph F and a colored graph Γ obtained by coloring the edges of F with at most r colors, we consider the number of r-edge-colorings of a larger graph G that avoids the *color pattern* of Γ. Here, a pattern $\widehat{F}$ of a graph F is defined as any partition of the edge set of F, and the pattern given by a coloring Γ is simply the pattern induced by the color classes. Notice that in a pattern we ignore the name of the colors. We let $c_{r,\widehat{F}}(G)$ denote the number or r-colorings of G which contain no k-vertex subgraph whose color pattern is isomorphic to the one of $\widehat{F}$; naturally, the quantity $c_{r,\widehat{F}}(n)$ is the maximum of $c_{r,\widehat{F}}(G)$ over all n-vertex graphs. We say that a coloring that avoids the pattern of $\widehat{F}$ is $\widehat{F}$-free. When the context is clear we omit the subscripts in $c_{r,\widehat{F}}(G)$ and also refer to an $\widehat{F}$-free r-coloring simply as a *good coloring*. Also, a graph G on n vertices is called $(r, \widehat{F})$-extremal (or simply extremal), when $c_{r,\hat{F}}(G) = c_{r,\hat{F}}(n)$.

We note that Balogh [2] had also considered a multicolored variant of the original Erdős-Rothschild problem. Given F, Γ and G as before, he considered the number $C_{r,\Gamma}(G)$ of r-colorings of G which do not contain a copy of F colored *exactly* as Γ (that is, in his version, we were not allowed to permute the colors). Observe that, if $\widehat{F}$ is the pattern given by Γ $c_{r,\widehat{F}}(G) \leq C_{r,\Gamma}(G)$, but the notions of these two quantities are different. For example, consider the case where Γ is a coloring of F that uses only one of the r colors, say "blue". In this case, $c_{r,\widehat{F}}(G)$ counts the number of colorings of G that avoids monochromatic copies of F, agreeing with the previous definition of $c_{r,F}(G)$, while $C_{r,\Gamma}(G)$ is the number of colorings of G which does not contain a blue copy of F (but may contain monochromatic copies of F in other colors). As another example, if one considers r-colorings of G, but the coloring of Γ uses at most $r-1$ of the colors, then the complete graph K_n is always extremal for $C_{r,\Gamma}(n)$, as the missing color may be used for any edge and hence may be used to extend colorings of any n-vertex graph G to colorings of K_n. However, colorings may not always be extended in this way in the case where we want to avoid color patterns, that is, when we are searching for the extremal graphs of $c_{r,\widehat{F}}(n)$.

Balogh [2] proved that in the case where $r=2$ and Γ is a 2-coloring of a clique K_k that uses both colors then $C_{2,\Gamma}(n) = 2^{\mathrm{ex}(n,K_k)}$ for n large enough, so the Turán graph $T_{k-1}(n)$ allows the maximum number of 2-colorings with no copy of Γ. (Note that this implies $c_{2,\widehat{F}}(n) = 2^{\mathrm{ex}(n,F)}$ for any pattern of K_k with two classes.) However, the picture changes if we consider 3-colorings with no rainbow triangles (pattern R_0 in Figure 1): Balogh also observed that, if we color the complete graph K_n with any two of the three colors available, there is no rainbow copy of K_3, which gives at least $3 \cdot 2^{\binom{n}{2}} - 3 \gg 3^{\mathrm{ex}(n,K_3)} = 3^{n^2/4+o(n^2)}$ distinct colorings avoiding rainbow triangles. (As usual, we say that two positive functions g, f satisfy $g(n) \ll f(n)$ if $\lim_{n\to\infty} g(n)/f(n) = 0$.)

In this paper, we focus on the case where $r \geq 3$ and the pattern is given by any edge-coloring of a *clique* that is not monochromatic. The paper has two parts which use very different techniques. In the first part, corresponding to Section 2, we shall use some ideas from the so called Zykov's symmetrization [24] (which also yields one of the classical proofs of Turán's theorem), together with Hölder's Inequality for a certain vector space, to prove a general result that works for arbitrary patterns (including the monochromatic one). First we show the following:

Theorem 1.1. *Let $\widehat{F}_k$ be any r-coloring of K_k. For every natural n, there exists a complete multipartite graph on n vertices which is $(r, \widehat{F}_k)$-extremal.*

Very recently, Pikhurko, Staden and Yilma [16] have obtained a similar result, albeit for a different extension of the original problem about monochromatic patterns (their forbidden patterns are still only monochromatic cliques, but they forbid cliques of different sizes for different colors).

In addition, we also proved that whenever the pattern is non-monochromatic and is different than two particular small patterns, then *every* extremal graph is a complete multipartite one.

Theorem 1.2. *Let $r \geq 2$ and $k \geq 3$ be given and let $\widehat{F}_k$ be an r-coloring of K_k which is not monochromatic and is different from the pattern T_0. Also assume that if $r=2$ then $\widehat{F}_k$ is different from the pattern P_2 (see Figure 1). Then* every *$(r, \widehat{F}_k)$-extremal graph is a complete multipartite graph.*

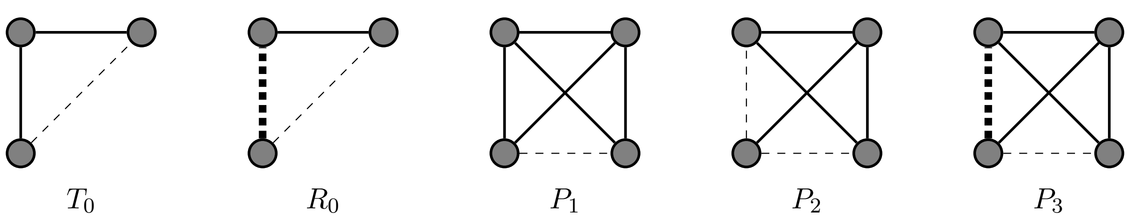

Figure 1: Some special patterns of colorings: T_0, P_1, P_2, use two colors, and R_0 and P_3 use three colors.

We remark that when $r=2$, the previously mentioned results [1, 2, 23] for $C_{2,\widehat{F}_k}(n)$ already imply that $c_{2,\widehat{F}_k}(n) = 2^{\mathrm{ex}(n,K_k)}$ for every $k \geq 3$ and every 2-coloring $\widehat{F}_k$ of the complete graph

K_k. In particular, in the case where $\widehat{F}_k = P_2$ and $r = 2$, our proof of Theorem 1.2 does not work, but we already know the exact optimum. Furthermore, if $r = 3$ and $\widehat{F}_3$ is the pattern T_0 in Figure 1, then our main theorem in Section 5 implies that the (only) extremal graph is the Turán graph. We believe that the conclusion in Theorem 1.2 actually works for any pattern given by a coloring of a clique.

A strong implication of Theorem 1.2 is that, in order to find any extremal graph of the pattern in the statement, we only have to find the number of vertices in each class that maximizes the number of colorings. We believe that if the pattern has some symmetry, then the number of vertices in each class must be the same. However, we have no indication that this must be true for all patterns. As a matter of fact, if we do not require the forbidden graph to be complete, there are instances where the extremal graph is complete multipartite, but the classes are not equitable, see [11, 7] in the case of matchings, or where the extremal graph is not even complete multipartite, see [9].

The exact extremal graph is known only for a very small values of r and very particular patterns. In most cases, when we do know the exact extremal graph for $c_{r,\widehat{F}}(n)$, it happens that we have equality in (1) and the extremal is the graph on n vertices and $\mathrm{ex}(n, F)$ edges. Pikhurko and Yilma [17] have found the exact extremal graph for two cases where we do not have equality in (1): when $r = 4$ and $\widehat{F}$ is either a monochromatic K_3 or K_4. Of course, such extremal graphs for K_t, for both $t = 3$ and $t = 4$, are still complete multipartite graphs (in fact, they are also Turán graphs, but different from $T_{t-1}(n)$), as our result in Theorem 1.2.

In the second part of the paper, we focus on the case where $r = 3$ and try to get more precise results for a specific family of patterns. We extend to multicolored patterns the method of Alon, Balogh, Keevash and Sudakov [1] (see also [2]), which uses Szemerédi's Regularity Lemma. Our original motivation was only to look at the pattern T_0 (the two-colored triangle) and R_0 (the rainbow triangle) in Figure 1. We conjecture that the extremal graph for R_0 is the complete graph (when $r = 3$). In Section 4, our main result (Theorem 4.4) is an approximate version of this which says that an extremal graph is an almost complete graph (in two different ways). For the particular case where the colored graph is K_n, we give a very short proof (by induction) that the number of colorings of it avoiding a rainbow triangle is at most $\frac{3}{2}(n-1)! \cdot 2^{\binom{n-1}{2}}$. It came recently to our attention (through personal communication) that results of V. Falgas-Ravry, K. O'Connell, J. Stromberg, and A. Uzzell involving the Entropy Method, Graph Limits and the Containers Method lead to a weaker bound of the form $2^{(1+o(1))\binom{n}{2}}$.

Finally, in Section 5, we prove that for any pattern $\widehat{F}_k$ (generated by a coloring of K_k) that satisfies a certain stability condition the (only) extremal graph for $c_{r,\widehat{F}_k}(n)$ is the Turán graph $T_{k-1}(n)$, for each n large enough. Afterwards, we show that such stability is satisfied by patterns that use only two colors and one of which induces a graph of small Ramsey number, which includes the pattern R_0. Given a graph F, the *Ramsey number* $R(F, F)$ is the smallest number ℓ such that any edge-coloring of the complete graph K_ℓ with two colors contains a monochromatic copy of F. Together, these results add up to the following main theorem.

Theorem 1.3. *Let $k \geq 3$ and let $\widehat{F}$ be a pattern of K_k with two classes, one of which induces a graph J such that $R(J, J) \leq k$. Then, for n sufficiently large, an n-vertex graph G satisfies $c_{3,\widehat{F}}(G) = c_{3,\widehat{F}}(n)$ if and only if G is isomorphic to the Turán graph $T_{k-1}(n)$.*

Recently, there has also been progress in finding graphs that admit the largest number of r-colorings avoiding some pattern of a complete graph for $r \geq 4$ colors. Typically, the results obtained focus on rainbow patterns of K_k, that is, patterns where all edges are assigned different colors, and show that the Turán graph $T_{k-1}(n)$ is optimal for large n as long as $r \geq r_0(k)$. For instance, in the case $k = 3$, this is known for $r_0 = 5$ (see [12]). In [12], the authors also extend the general method of [1] to multicolored patterns, but the results in the first part allow us to shorten it slightly.

2. Results that hold for every coloring pattern $\widehat{F}_k$ of a clique K_k.

For this section, $r \geq 2$ and $k \geq 3$ are natural numbers and $\widehat{F}_k$ is any r-coloring of a complete graph K_k.

For a vector $\vec{x}$ indexed by a set T, we will denote by $x(t)$ the value of x at coordinate t, where $t \in T$. We will use $\|\vec{x}\|_p$ to denote the ℓ_p-norm of $\vec{x}$, so for $p \in (0, \infty)$ we have

$$\|x\|_p = \left(\sum_{t \in T} |x(t)|^p \right)^{1/p}.$$

Moreover, for a sequence of vectors $x_1, \ldots, x_s$, each indexed by T, we will denote their pointwise product by $\prod_{k=1}^{s} \vec{x}_k$, that is, the vector y such that for each $t \in T$ we have $y(t) = \prod_{k=1}^{s} x_k(t)$.

Definition 2.1. *If H is a subgraph of a graph G and $\widehat{H}$ is an $\widehat{F}_k$-free r-coloring of H, we denote by $c_{r,\widehat{F}_k}(G \mid \widehat{H})$ the number of ways to r-color the edges in $E(G) - E(H)$ in such a way that the resulting coloring is still $\widehat{F}_k$-free. For a single vertex $v \in V(G) - V(H)$, we use the notation $c_{r,\widehat{F}_k}(v, \widehat{H})$ for the number of ways to r-color the edges from v to $V(H)$ (again avoiding $\widehat{F}_k$). We also define $\vec{v}_H$ as the vector indexed by the $\widehat{F}_k$-free r-colorings of H, whose coordinate corresponding to a coloring $\widehat{H}$ is given by $\vec{v}_H(\widehat{H}) = c_{r,\widehat{F}_k}(v, \widehat{H})$.*

We have the following immediate proposition.

Proposition 2.2. *If H is an induced subgraph of G such that $S = V(G) - V(H)$ is an independent set in G, and $\widehat{H}$ is an $\widehat{F}_k$-free r-coloring of H, then*

$$c(G \mid \widehat{H}) = \prod_{v \in S} c(v, \widehat{H}).$$

Proof. It follows trivially from the fact that there is no K_k that contains two vertices of S and therefore the choice of colors of the edges incident to a vertex of S does not affect the colors of edges incident to other vertices of S. $\square$

We will need the inequality below, known as the Generalized Hölder's Inequality (stated here for the particular case of the counting measure on a finite set). For a more general version see the book [22] (chapter 8, exercise 6).

Lemma 2.3 (Hölder's Inequality). *Assume that $r \in (0, \infty)$ and $p_1, p_2, \ldots, p_s \in (0, \infty]$ are such that*

$$\sum_{k=1}^{s} \frac{1}{p_k} = \frac{1}{r},$$

and let $\vec{x_1}, \ldots, \vec{x_s}$ be complex-valued vectors indexed by a common set T. We have

$$\left\| \prod_{k=1}^{s} \vec{x}_k \right\|_r \leq \prod_{k=1}^{s} \|\vec{x}_k\|_{p_k}.$$

Furthermore, equality happens above if and only if for every $i, j \in [s]$ there is $\alpha_{i,j}$ such that for every $t \in T$ we have

$$|\vec{x}_i(t)|^{p_i} = \alpha_{i,j} \cdot |\vec{x}_j(t)|^{p_j}.$$

Remark. One may easily check that $\alpha_{i,j} = \frac{\left(\|\vec{x}_i\|_{p_i}\right)^{p_i}}{\left(\|\vec{x}_j\|_{p_j}\right)^{p_j}}$.

We will actually use it only in the following particular form.

Corollary 2.4. *Let $\vec{x}_1, \ldots, \vec{x}_s$ be complex-valued vectors indexed by the same set. We have*

$$\left\| \prod_{k=1}^{s} \vec{x}_k \right\|_1 \leq \prod_{k=1}^{s} \|\vec{x}_k\|_s .$$

Furthermore, equality happens if and only if for every $i, j \in [s]$ the vector $(|x_i(t)|)_{t\in T}$ (whose entries are the absolute values of those in $\vec{x}_i$) is a multiple of $(|x_j(t)|)_{t\in T}$.

Proof. Take $r = 1$ and, for $1 \leq i \leq s$, take $p_i = s$ in the statement of Lemma 2.3. The equality condition, also follows from the equality condition in Lemma 2.3. □

Remark 2.5. When $s = 2$, the inequality in Corollary 2.4, is equivalent to the Cauchy-Schwartz inequality: $\langle \vec{x}_1, \vec{x}_2 \rangle \leq \|\vec{x}_1\| \, \|\vec{x}_2\|$.

Definition 2.6. *We say that two vertices are* twins *if they are non-adjacent and have the same neighborhood.* Cloning *a vertex v of a graph G means to create a new graph $\widetilde{G}$ whose vertex set is $V(G) \cup \{\widetilde{v}\}$ where $\widetilde{v}$ is a new vertex which is a twin of v.*

For the next lemma we consider the following operation: take an independent set S of a graph G, select a particular vertex $v \in S$, delete all vertices in $S - v$ and add $|S| - 1$ new twins of v. The result is a new graph which has at least as many good colorings as G.

Lemma 2.7. *Let $\widehat{F}_k$ be any r-coloring of K_k. Let G be a graph on n vertices, $S \subset V(G)$ be an independent set with $s = |S|$, $H = G - S$, and $A = V(G) - S$. There exists a vertex $v \in S$ with the following property: if we construct the graph $\widetilde{G}$ with $V(\widetilde{G}) = V(H) \cup \widetilde{S}$, where $\widetilde{S}$ is an independent set on s vertices, each of which is a twin of v, and $\widetilde{G}[A] = G[A]$, then:*

(a) $c_{r,\widehat{F}_k}(\widetilde{G}) \geq c_{r,\widehat{F}_k}(G)$;

(b) If G is $(r, \widehat{F}_k)$-extremal, then for every $u, w \in S$ we must have $\vec{u}_H = \vec{w}_H$.

Proof. Let S be any independent set in G, and let $H = G - S$. For each $u \in S$, consider the vector $\vec{u}_H$ as in Definition 2.1. By Proposition 2.2, the total number of $\widehat{F}_k$-free r-colorings of G is

$$c(G) = \sum_{\widehat{H}} c(G \mid \widehat{H}) = \sum_{\widehat{H}} \prod_{u\in S} c(u, \widehat{H}) = \left\| \prod_{u\in S} \vec{u}_H \right\|_1 ,$$

where the sums are taken over all possible $\widehat{F}_k$-free r-colorings $\widehat{H}$ of H. (For the last equality we also used that every coordinate of $\vec{u}_H$ is non-negative).

Let v be a vertex in S for which $\|\vec{v}_H\|_s$ is maximum. This fact, together with Hölder's Inequality (Corollary 2.4), gives us:

$$\left\| \prod_{u\in S} \vec{u}_H \right\|_1 \leq \prod_{u\in S} \|\vec{u}_H\|_s \leq \|\vec{v}_H\|_s^s . \tag{2}$$

On the other hand, for the graph $\widetilde{G}$ defined in the statement of this lemma, we have:

$$c(\widetilde{G}) = \sum_{\widehat{H}} c(v, \widehat{H})^s = \|\vec{v}_H\|_s^s .$$

Therefore, $c(\widetilde{G}) \geq c(G)$.

To prove part (b), assume G is extremal and $\widetilde{G}$ is as above. Since $c(\widetilde{G}) \geq c(G)$, we must have $c(\widetilde{G}) = c(G)$. Therefore, we must also have equality in both inequalities in (2). From the second one, it follows that for every $u \in S$, we must have $\|\vec{u}_H\|_s = \|\vec{v}_H\|_s$. From the first one, where we used Corollary 2.4, the equality condition in such corollary, together with the fact that all our vectors have only non-negative entries, and the fact that $\|\vec{u}_H\|_s = \|\vec{v}_H\|_s$, implies that $\vec{u}_H = \vec{v}_H$. And it follows trivially, that for every $u, w \in S$ we must have $\vec{u}_H = \vec{w}_H$. $\square$

Corollary 2.8. *If G is an $(r, \widehat{F}_k)$-extremal graph, and $u, v \in V(G)$ are any non-adjacent vertices, then deleting v and cloning u produces a graph that is also extremal.*

Proof. Since G is extremal, by Lemma 2.7-(b) with $S = \{u, v\}$ and $G_{uv} = G - \{u, v\}$, we must have $\vec{u}_{G_{uv}} = \vec{v}_{G_{uv}}$, therefore replacing v by a twin of u (or u by a twin of v) does not change the number of colorings of the graph. $\square$

By repeatedly applying Corollary 2.8 above, we can easily show that *there exists* a complete multipartite graph on n vertices which is $(r, \widehat{F}_k)$-extremal. Although this is a direct consequence of Corollary 2.8, we spell out the details. On the other hand, showing that (for non-monochromatic patterns) *every* extremal is a complete multipartite graph will require more work.

Theorem 1.1. *Let $\widehat{F}_k$ be any r-coloring of K_k. For every natural n, there exists a complete multipartite graph on n vertices which is $(r, \widehat{F}_k)$-extremal.*

Proof of Theorem 1.1. Let G be any $(r, \widehat{F}_k)$-extremal graph on n vertices. We will build a sequence of extremal graphs, each on n vertices, say $G_0, G_1, \ldots, G_t$, where $G_0 = G$, and G_t is a complete multipartite graph. We do it in such a way that, for $i \geq 1$, we have $V(G_i) = S_1 \cup \cdots \cup S_i \cup R_i$, where for every $j \in \{1, \ldots, i\}$, the set S_j is an independent set and every vertex in S_j is adjacent to every vertex outside S_j (including those in R_i), but we have no control of the edges inside R_i. It will also hold that $R_t \subset R_{t-1} \subset \cdots \subset R_1 \subset V(G)$, and R_t will be independent.

To simplify the notation, we also define $R_0 = V(G_0) = V(G)$. Assume that we have constructed G_i, for some $i \geq 0$. If R_i is an independent set, we have found a complete multipartite graph which is extremal, so we can set $t = i$ and stop. Otherwise, let v_i be any vertex of R_i that has a neighbor in R_i. Note that, by the definition of G_i, all non-neighbors of v_i belong to R_i. Let $\overline{d}_i$ be the number of non-neighbors of v_i. We can obtain G_{i+1} applying Corollary 2.8 successively $\overline{d}_i$ times, deleting each non-neighbor of v_i and adding twins of v_i (one by one). Let S_{i+1} be the set formed by v_i and its new twins and let R_{i+1} to be the set of neighbors of v_i in R_i. Observe that R_{i+1} is strictly smaller than R_i since it does not contain v_i. It is also important to notice that, at every step when we use Corollary 2.8 we apply it to the whole graph G_i and not only to $G_i[R_i]$. $\square$

Observe that in the proof of Theorem 1.1 we may select v_i as the vertex with the largest degree in R_i. By doing this, we obtain, starting from an extremal graph G, a complete multipartite graph that has at least as many edges as G. In the next lemma, we show that if G is not complete multipartite itself, we can find another complete multipartite extremal graph by only deleting edges of G.

Lemma 2.9 (Edge deletion lemma)**.** *Let $\widehat{F}_k$ be any r-coloring of graph K_k and $r \geq 2$ be a natural number. Let G be an $(r, \widehat{F}_k)$-extremal graph. For any u, v, w such that $uv, uw \notin E(G)$ and $vw \in E(G)$, if we delete the edge vw, then the resulting graph is still extremal.*

Proof. Let G be a graph as in the statement. Fix vertices u, v, w such that $uv, uw \notin E(G)$ and $vw \in E(G)$ (this implies that G is not a complete multipartite graph).

Let $H = G - \{u, v, w\}$, and $H^x = G[V(H) \cup x]$ for $x \in \{u, v, w\}$. Let G' be the graph obtained from G by deleting the edge vw (but not the vertices u or v), and let G^* be the graph obtained from H^u by adding another two clones of u, say u_1 and u_2. By Corollary 2.8, the graph G^* is also extremal, as we may first apply the replacement operation to the pair u, v (deleting v and adding u_1) and apply it again to the pair u, w. Therefore, $c(G) = c(G^*)$.

Applying Proposition 2.2 to G^* with $S = \{u, u_1, u_2\}$, we have

$$c(G^*) = \sum_{\widehat{H}} c(G^* \mid \widehat{H}) = \sum_{\widehat{H}} c(u, \widehat{H})^3 = \|\vec{u}_H\|_3^3 \,,$$

where the sum is taken over all $\widehat{F}_k$-free r-colorings of H.

Observe that, with an analogous computation, if we start from H and add three clones of w instead of u, the resulting graph has $\|\vec{w}_H\|_3^3$ good colorings. But we do not know if such graph is extremal, so we have only

$$\|\vec{w}_H\|_3^3 \leq \|\vec{u}_H\|_3^3 \,. \tag{3}$$

On the other hand, since there are no edges from u to $\{v, w\}$, we can compute $c(G)$ as follows:

$$\begin{aligned} c(G) &= \sum_{\widehat{H}} \Big(c(u, \widehat{H}) \cdot c(G - u \mid \widehat{H}) \Big) \\ &= \sum_{\widehat{H}} \left(c(u, \widehat{H}) \cdot \left(\sum_{\widehat{H^w} \mid \widehat{H}} c(v, \widehat{H^w}) \right) \right) . \end{aligned} \tag{4}$$

Here, the inner sum is taken over the good colorings of H^w that extend a given good coloring of H, that is, over the colorings of the edges from w to H, for which the resulting coloring is good. By Lemma 2.7-(b), since G is extremal and $uv \notin E(G)$, we have $\vec{v}_{H^w} = \vec{u}_{H^w}$, that is $c(v, \widehat{H^w}) = c(u, \widehat{H^w})$ for every $\widehat{H^w}$. Finally, note that $c(u, \widehat{H^w})$ does not depend on the colors of the edges from w to H, so $c(u, \widehat{H^w}) = c(u, \widehat{H})$. Therefore,

$$c(G) = \sum_{\widehat{H}} \left(c(u, \widehat{H}) \left(\sum_{\widehat{H^w} | \widehat{H}} c(u, \widehat{H}) \right) \right) \tag{5}$$

$$= \sum_{\widehat{H}} \left(c(u, \widehat{H}) c(u, \widehat{H}) \sum_{\widehat{H^w} | \widehat{H}} 1 \right) \tag{6}$$

$$= \sum_{\widehat{H}} c(u, \widehat{H})^2 c(w, \widehat{H}) \tag{7}$$

$$\leq \|\vec{u}_H\|_3 \, \|\vec{u}_H\|_3 \, \|\vec{w}_H\|_3 \tag{8}$$

$$\leq \|\vec{u}_H\|_3^3 \, . \tag{9}$$

Notice that to get (8) we used Hölder's Inequality (Corollary 2.4), and (9) follows from (3). Finally, since $c(G) = \|\vec{u}_H\|_3^3$, we must have equality in both (8) and (9), which in turn leads to $\|\vec{u}_H\|_3 = \|\vec{w}_H\|_3$. The equality condition in Lemma 2.3 implies that $\vec{u}_H = \vec{w}_H$. Analogously, $\vec{u}_H = \vec{v}_H$. It follows that

$$c(G^*) = \sum_{\widehat{H}} c(u, \widehat{H}) c(v, \widehat{H}) c(w, \widehat{H}) = c(G').$$

$\square$

Finally, we use Lemma 2.9 to prove our main result of this section, which we restate below.

Theorem 1.2. *Let $r \geq 2$ and $k \geq 3$ be given and let $\widehat{F}_k$ be an r-coloring of K_k which is not monochromatic and is different from the pattern T_0. Also assume that if $r = 2$ then $\widehat{F}_k$ is different from the pattern P_2 (see Figure 1). Then* every *$(r, \widehat{F}_k)$-extremal graph is a complete multipartite graph.*

Proof. Let $\widehat{F}_k$ be an r-coloring as in the statement. Suppose that there exists an $(r, \widehat{F}_k)$-extremal graph G which is not a complete multipartite graph. Let u, v, w, H, H^v, and H^w be defined as in the proof of Lemma 2.9. At the end of the proof, we concluded $\vec{u}_H = \vec{w}_H = \vec{v}_H$, so for every coloring $\widehat{H}$ of H we have $c(u, \widehat{H}) = c(w, \widehat{H}) = c(v, \widehat{H})$. We also noticed that, for every extension of $\widehat{H}$ to a coloring $\widehat{H^w}$, we have $c(u, \widehat{H^w}) = c(u, \widehat{H})$.

Now note that, since u and v are not adjacent, by Lemma 2.7-(b), we have $\vec{u}_{H^w} = \vec{v}_{H^w}$, that is, $c(u, \widehat{H^w}) = c(v, \widehat{H^w})$ for every $\widehat{H^w}$. From the previous equalities, it follows that, for every $\widehat{F}_k$-free extension $\widehat{H^w}$ of $\widehat{H}$, we must have

$$c(v, \widehat{H^w}) = c(v, \widehat{H}). \tag{10}$$

Our goal here is to get a contradiction from this fact (which implies that such G cannot exist). We only need to find an r-coloring of H and an extension of it to H^w, which is $\widehat{F}_k$-free and such that equation (10) does not hold. We will split the proof into cases, depending on the pattern of $\widehat{F}_k$. In each case we proceed as follows. We fix a particular good coloring $\widehat{H^w}$ of H^w and consider the coloring $\widehat{H}$ induced by it in H. Let $\mathcal{H}(v)$ and $\mathcal{H}^w(v)$ denote the set of $\widehat{F}_k$-free extensions of $\widehat{H}$ to H^v and of $\widehat{H^w}$ to $G - u$, respectively. To find a contradiction to (10), we show that there is an injective mapping $\phi\colon \mathcal{H}(v) \to \mathcal{H}^w(v)$ that is not surjective.

We say that a coloring of $\widehat{F}_k$ is *almost monochromatic* if it is not monochromatic and there exists a vertex $x \in F_k$ such that all edges not incident to x have the same color, say color 1, and there is at least one edge incident to x that is also of color 1. We call such x the *special* vertex.

The remainder of the proof splits the analysis into four cases. Figure 2 illustrates how colorings are extended in each case.

Case 1: $\widehat{F}_k$ is *not* almost monochromatic. Let $\widehat{H^w}$ be the coloring that assigns color blue to all edges of H^w, so that $\widehat{H}$ is a blue coloring of H. To define the injective mapping $\phi\colon \mathcal{H}(v) \to \mathcal{H}^w(v)$, for any extension of $\widehat{H}$ to H^v, consider the same extension of $\widehat{H^w}$ to the edges between v and H and assign blue to the edge vw. By definition of good coloring, there is no $\widehat{F}_k$ in the extension to H^v or in H^w, so any copy of $\widehat{F}_k$ must be induced by a set that contains vw. However, any such set, induces a coloring that is almost monochromatic (in which v plays the role of the special vertex x). On the other hand, consider the coloring of $G - u$ where all edges are blue, with the exception of the edge vw, which is colored red. Any pattern contained in this coloring is either monochromatic or almost monochromatic, and therefore is different from $\widehat{F}_k$. However, it is not in the image of ϕ.

Case 2: $\widehat{F}_k$ *is* almost monochromatic and is different from the patterns T_0, P_1, P_2, P_3 of Figure 1. Let $\widehat{H^w}$ be such that all edges inside H are blue and the ones from w to H are red. To define ϕ, for any good coloring that extends $\widehat{H}$ to the edges between v and H, extend it by coloring vw with red. As before, we only need to check that any pattern that contains the edge vw is not equal to $\widehat{F}_k$. Notice that here we must have $k \geq 4$ (as $\widehat{F}_k$ is almost monochromatic and different from T_0). Suppose that there is an almost monochromatic pattern that contains vw. Note that it must contain exactly two vertices of H, one of which is the image of the special vertex x. Because all edges in $F_k - x$ have the same color, $\widehat{F}_k$ must be equal to P_1, P_2 or P_3, a contradiction. To see that ϕ is not surjective, let all edges from v to H be red and the edge vw be blue. It is easy to check that the only pattern which is almost monochromatic and is contained in this coloring is T_0.

Case 3: $\widehat{F}_k$ *is* P_1 or P_3, given in Figure 1. Let $\widehat{H^w}$ be such that all edges inside H are blue and the ones from w to H are red. To define ϕ, given a good coloring that extends $\widehat{H}$ to the edges between v and H, extend it to $G - u$ by coloring vw with blue. It is easy to see that this cannot produce P_1 or P_3 using vw. Again, this function ϕ is not surjective, as we may color all edges between v and H with blue and let vw be red.

Case 4: $\widehat{F}_k$ *is* P_2, given in Figure 1. In this case we assume $r \geq 3$. Let $\widehat{H^w}$ be such that all edges inside H are blue and the ones from w to H are red. To define ϕ, given a good coloring that extends $\widehat{H}$ to the edges between v and H, extend it to $G - u$ by coloring vw with a third color, say green. Clearly, any four vertices containing v and w induce a pattern that uses at least three colors, and thus is not equal to P_2. Note that the extension of $\widehat{H^w}$ such that all edges from v to H^w are red does not contain the pattern P_2, so that ϕ is not surjective.

□

Remark 2.10. Note that if $\widehat{F}_3$ is a rainbow coloring of K_3, then it is treated in Case 1 of Theorem 1.2. The proof that we gave here does not work for monochromatic pattern simply because our colorings of H^w always contain monochromatic cliques.

3. The case of 3-colorings - Auxiliary results

In the remainder of this paper, we shall only be concerned with colorings with three colors. In Sections 4 and 5, our proofs will be based on the Regularity Method of Szemerédi together with some stability results. Here, we give the necessary definitions and state the main results that we shall use.

Showing that ϕ is injective

A coloring that is not in the image of ϕ

Case 1: Avoid patterns which are not almost monochromatic, neither monochromatic.

Case 2: Avoid patterns which are almost monochromatic, except T_0, P_1, P_2, P_3.

Case 3: Avoid P_1 and P_3.

Case 4: Avoid P_2, for $r \geq 3$.

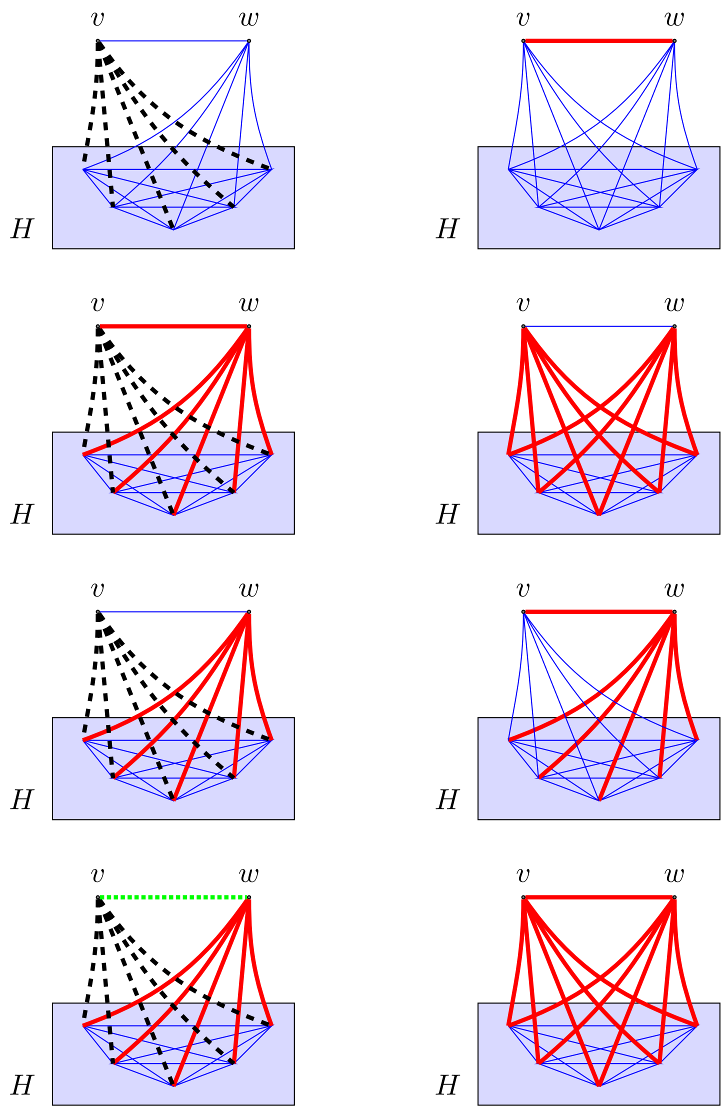

Figure 2: How to color and to extend a coloring in each case.

Given two disjoint non-empty sets of vertices X and Y of a graph G, we let $E(X,Y)$ denote the set of edges with one end in X and the other one in Y. We also set $e(X,Y) = |E(X,Y)|$ and let $d(X,Y) = \frac{e(X,Y)}{|X||Y|}$ denote the edge density between X and Y.

Definition 3.1. *Let $G = (V,E)$ be a graph and let $0 < \varepsilon \leq 1$. We say that a pair (A,B) of two disjoint subsets of V is* ε-regular *(with respect to G) if*

$$|d(A',B') - d(A,B)| < \varepsilon$$

holds for any two subsets $A' \subset A$, $B' \subset B$ with $|A'| > \varepsilon|A|$, $|B'| > \varepsilon|B|$.

Definition 3.2. *Given a graph $G = (V,E)$, a partition $V = V_1 \cup \ldots \cup V_t$ is called* ε-regular *(with respect to G) if:*

(a) $||V_i| - |V_j|| \leq 1$ for every $i,j \in \{1,\ldots,t\}$, and

(b) (V_i,V_j) is ε-regular for all but at most εt^2 of the pairs (V_i,V_j) where $i \neq j$.

In our proofs, we shall make use of a colored version of the Szemerédi Regularity Lemma [20] stated in [13].

Lemma 3.3. *For every $m, \varepsilon > 0$ and integer r, there exist n_0 and M such that, if the edges of a graph G of order $n \geq n_0$ are r-colored, say $E(G) = E_1 \cup \cdots \cup E_r$, then there is a partition of the vertex set $V(G) = V_1 \cup \cdots \cup V_t$ with $m \leq t \leq M$ which is ε-regular simultaneously with respect to all graphs $G_i = (V,E_i)$ for $i = 1,\ldots,r$.*

A partition as in Lemma 3.3 will be called a *multicolored ε-regular partition*. Given such a partition and given a color $\sigma \in [r]$, we can define a *cluster graph* associated with color σ as follows. Given $\eta > 0$, the graph $R_\sigma = R_\sigma(\eta)$ is defined on the vertex set $[t]$ so that $\{i,j\} \in E(R_\sigma)$ if and only if (V_i,V_j) is an ε-regular pair with edge density at least η with respect to the subgraph of G induced by the edges of color σ.

We may also define the *multicolored cluster graph* R associated with this partition: the vertex set is $[t]$ and $e = \{i,j\}$ is an edge of R if $e \in E(R_\sigma)$ for some $\sigma \in [r]$. Each edge e in R is assigned the list of colors $L_e = \{\sigma \in [r] \mid e \in E(R_\sigma)\}$. Given a colored graph Γ, we say that a multicolored cluster graph R contains Γ if R contains a copy of the underlying graph of Γ such that the color of each edge (with respect to Γ) is contained in the list of the corresponding edge in R. More generally, if F is a graph with color pattern $\widehat{F}$, we say that R contains $\widehat{F}$ if it contains some colored copy of F with pattern $\widehat{F}$.

One of the main advantages of considering cluster graphs are embedding results that ensure that some substructure found within a cluster graph can also be found in the original graph. In the present work, the following embedding result will be particularly useful. It is stated in terms of 3-colorings because of our setting, but the same statement would hold for r colors. The proof is quite standard and follows the arguments in the proof of [13, Theorem 2.1], which is known as the Key Lemma.

Lemma 3.4. *For every $\eta > 0$ and every positive integer k, there exist $\varepsilon = \varepsilon(\eta,k) > 0$ and a positive integer $n_0(\eta,k)$ with the following property. Suppose that G is a 3-edge colored graph on $n > n_0$ vertices with a multicolored ε-regular partition $V = V_1 \cup \cdots \cup V_t$ which defines the multicolored cluster graph $R = R(\eta)$. Let F be a fixed k-vertex graph with a prescribed color pattern $\widehat{F}$. If R contains $\widehat{F}$, then the graph G also contains $\widehat{F}$.*

The following classical stability result will also be used in our proofs.

Theorem 3.5. *[3, 19] For every $\alpha > 0$ there exist $\beta > 0$ and n_0 such that any K_k-free graph on $n \geq n_0$ vertices with at least $\mathrm{ex}(n, K_k) - \beta n^2$ edges has a partition $V = V_1 \cup \cdots \cup V_{k-1}$ of the vertex set with $\sum e(V_i) < \alpha n^2$.*

We will also need the entropy function, which we will denote by $H(x)$, and is defined as $H(x) = -x \log_2(x) - (1-x)\log_2(1-x)$, for $0 < x < 1$. It will be useful for the well known estimate

$$\binom{a}{xa} \leq 2^{H(x)a}. \tag{11}$$

Note that $\lim_{x \to 0^+} H(x) = 0$.

4. 3-colorings avoiding a rainbow triangles

Throughout this section we let $\widehat{F}_3$ be a 3-colored rainbow K_3, that is, one in which all edges have different colors. Here, we will use the Regularity Method to show that every $(3, \widehat{F}_3)$-extremal graph is an 'almost complete' graph. Recall that we already know that it is a complete multipartite graph.

For reasons that will be clear later, we will need to solve the problem of maximizing the value $w(G)$ defined below, over the set of graphs with a given number of vertices.

Definition 4.1. *Given a graph G, let $w : E(G) \to \{2, 3\}$ be the function that gives weight 2 or 3 to the edges of G in such a way that every edge that belongs to some triangle gets weight 2 and all the remaining edges get weight 3. Define $w(G)$ to be the product of the weight of the edges of G.*

The following lemma tells us that for a given number of vertices, the value of $w(G)$ is maximal when G is a complete graph.

Lemma 4.2. *Given a graph G on t vertices, the function $w(G)$ defined above satisfies $w(G) \leq 2^{\binom{t}{2}}$.*

Our proof of Lemma 4.2 (which works for all values of t) is based, again, on Zykov's Symmetrization. The next lemma is a stronger result, but works only for large values of t. Since, in this article, we will only need results for large values of t, we postpone the proof of Lemma 4.2 to the appendix. Here, given a graph G, let $\bar{e}(G)$ be the number of edges in the complement $\overline{G}$ of G.

Lemma 4.3. *Let G be a graph on $t > 1000$ vertices. Attribute weights to the edges of G as in Definition 4.1. For $i \in \{2, 3\}$, let e_i be the number of edges of weight i and let $\bar{e} = \bar{e}(G)$. If $\bar{e} \leq t^2/4$, then $e_3 \leq (\sqrt{2} + 0.01)\bar{e}$ and $w(G) = 2^{e_2}3^{e_3} \leq 2^{\binom{t}{2}} 2^{-0.16\bar{e}}$. And trivially, if $\bar{e} > t^2/4$ then $w(G) \leq 3^{\binom{t}{2} - \bar{e}} < 3^{t^2/4} \ll 2^{\binom{t}{2}}$.*

Proof. Let G be a graph such that $\bar{e} \leq t^2/4$. We will double count the number of pairs (uv, ab) where uv is an edge of weight 3 of G and ab is an edge in the complement of G and $\{u, v\} \cap \{a, b\} \neq \emptyset$. Let T be the number of such pairs.

For every edge uv of weight 3, we must have $N(u) \cap N(v) = \emptyset$. Therefore, $d(u) + d(v) \leq t$. This implies that $\bar{d}(u) + \bar{d}(v) \geq 2(t-1) - t = t - 2$. Therefore, there are at least $t - 2$ edges ab of $\bar{G}$ which are incident with uv. This implies that $T \geq e_3(t-2)$.

Now, for each non-edge ab, we want to bound the number of edges of weight 3 which are incident with a or b. That is, if we denote these quantities $N_3(a)$ and $N_3(b)$, respectively, we want an upper bound on $|N_3(a)| + |N_3(b)|$ (noting that we are counting edges and not the number of vertices in $N_3(a) \cup N_3(b)$). We claim that for every $a \in V(G)$, we have that $|N_3(a)| \leq t/\sqrt{2} + 1$. In fact, since the edges ua where $u \in N_3(a)$ have weight 3, they do no belong to any triangle and therefore $N_3(a)$ is an independent set. This implies that

$$\frac{(|N_3(a)| - 1)^2}{2} \leq \binom{|N_3(a)|}{2} \leq \bar{e} \leq \frac{t^2}{4}.$$

Therefore, $|N_3(a)| \leq t/\sqrt{2} + 1$ as desired. The same bound holds for $|N_3(b)|$. It follows that $|N_3(a)| + |N_3(b)| \leq t\sqrt{2} + 2$. This implies that $T \leq (t\sqrt{2} + 2)\bar{e}$.

Comparing the upper bound and the lower bound for T, we have that: $(t\sqrt{2} + 2)\bar{e} \geq e_3(t-2)$, which implies $e_3 \leq (\sqrt{2} + \frac{2\sqrt{2}+2}{t-2})\bar{e} < (\sqrt{2} + 0.01)\bar{e}$. So we proved the first part of the statement. Now, we conclude that

$$2^{e_2}3^{e_3} = 2^{e_2+e_3+\bar{e}} \left(\frac{3}{2}\right)^{e_3} \frac{1}{2^{\bar{e}}} = 2^{\binom{t}{2}} 2^{\log_2(3/2)e_3 - \bar{e}} \leq 2^{\binom{t}{2}} 2^{(\log_2(3/2)(\sqrt{2}+0.01)-1)\bar{e}} \leq 2^{\binom{t}{2}} 2^{(0.84-1)\bar{e}}.$$

Finally, we note that the case where $\bar{e} > t^2/4$ is trivial (any remaining edge may have weight 3). □

The result below establishes two approximate results about $(3, \widehat{F}_3)$-extremal graphs. Recall that $c_{3,\widehat{F}_3}(K_n) \geq \binom{3}{2} \cdot 2^{\binom{n}{2}}$. So, Part (a) says that, for n large, the extremal number $c_{3,\widehat{F}_3}(n)$ is not much larger than the number of $(3, \widehat{F}_3)$-good-colorings of K_n. And Part (b) says that any large graph that has a chance of being extremal is a near complete graph.

Theorem 4.4. *The following hold for the rainbow triangle $\widehat{F}_3$.*

(a) For all $\delta > 0$ there exists n_0 such that, if G is a graph of order $n > n_0$, then $c_{3,\widehat{F}_3}(G) \leq 2^{(1+\delta)n^2/2}$.

(b) For all $\xi > 0$, there exists n_1 such that, if G is a graph of order $n > n_1$ and $c_{3,\widehat{F}_3}(G) \geq 2^{\binom{n}{2}}$, then $|E(G)| \geq \binom{n}{2} - \xi n^2$.

Proof. For part (a), fix $\delta > 0$ and consider $\eta > 0$ such that $2\eta + H(\eta) \leq \delta/4$. For this value of η, set n_0' and ε given by Lemma 3.4, where we further assume that $\varepsilon < \eta/8$. Let n_0'' and M be given by Lemma 3.3 with $m = 1/\varepsilon$.

Let $n_0 > \max\{n_0', n_0'', 1000\}$, where additionally the inequality (14) holds for all $n \geq n_0$ (for example, it is enough to have $M^n\, 2^{3M^2/2} < 2^{\delta n^2/4}$). Consider a graph $G = (V, E)$ with $n \geq n_0$ vertices. We want to bound the number of 3-colorings of G that do not have a rainbow triangle. Fix an arbitrary 3-edge-coloring of G with no rainbow triangle, and let $V_1 \cup \cdots \cup V_t$ be an ε-regular partition given by Lemma 3.3 associated with this coloring. Recall that $m \leq t \leq M$. In the following inequality, we use the symbol "$\ll$" (in a weak way) meaning "sufficiently smaller than". Our definitions lead to

$$\frac{1}{n_0} \ll \frac{1}{t} \leq \varepsilon \leq \frac{\eta}{4} \ll \delta. \tag{12}$$

With respect to the partition $V_1 \cup \cdots \cup V_t$, let R_1, R_2 and R_3 be the cluster graphs on the vertex set $\{1, \ldots, t\}$, with density $\eta/4 > 0$, associated with each color, and let R be the corresponding multicolored cluster graph.

First we bound the number of 3-edge colorings of G that could give rise to this particular partition and these cluster graphs. The number of edges that lie within some class of the partition is bounded above by $t\binom{n/t}{2} \leq n^2/(2t) \leq \varepsilon n^2/2 < \eta n^2/8$, while the number of edges joining a pair of vertices in classes that are not regular with respect to some color is at most $3\varepsilon t^2(n/t)^2 < 3\eta n^2/8$. There are also at most $3\eta/4 \cdot \binom{n}{2} \leq 3\eta n^2/8$ edges that join a pair of classes in which their color has density smaller than $\eta/4$. This adds to at most ηn^2 edges. There are at most $\binom{n^2}{\eta n^2}$ ways to choose this set of edges and they can be colored in at most $3^{\eta n^2}$ different ways.

For any pair (i, j) with $i < j$, the remaining edges joining V_i to V_j may be colored in at most $s_{i,j}$ ways, where $s_{i,j}$ is the number $(1 \leq s_{i,j} \leq 3)$ of cluster graphs amongst R_1, R_2 and R_3 for which $\{i, j\}$ is an edge. Since $e(V_i, V_j) \leq (n/t)^2$, there are at most $s_{i,j}^{n^2/t^2}$ ways to color these edges. Let E_s be the set of edges that appear in exactly s of the cluster graphs and denote $e_s = |E_s|$.

This discussion implies that the number of potential 3-edge colorings of G that could give rise to this vertex partition and these cluster graphs is at most

$$\binom{n^2}{\eta n^2} 3^{\eta n^2} (1^{e_1} 2^{e_2} 3^{e_3})^{n^2/t^2}. \tag{13}$$

Notice that, the above estimate works for any coloring pattern that we want to avoid, not only for the rainbow triangle. So, we shall use it again in the proof of Lemma 5.2, which is about a different pattern.

The term $3^{\eta n^2}$ may be replaced by the upper bound $2^{2\eta n^2}$, while the quantity $\binom{n^2}{\eta n^2} 3^{\eta n^2}$ may be bounded above by $2^{(H(\eta)+2\eta)n^2}$ because of (11).

Next, for an upper bound on $1^{e_1}2^{e_2}3^{e_3}$, note that this value may be obtained from R by giving weight i to the edges in E_i and multiplying the weights of the edges. Let R' be the subgraph of R obtained by deleting all the edges of E_1. Now, all the edges of R' have weight 2 or 3, and the product of the weights of its edges is still the same as in R. Notice that R' contains no triangle with an edge of weight 3, otherwise we could find a rainbow triangle in the multicolored cluster graph, which, in turn, by Lemma 3.4, would lead to a rainbow triangle in the original coloring. Therefore, $1^{e_1}2^{e_2}3^{e_3} \leq w(R')$, where w is defined as in Definition 4.1. By Lemma 4.2, we derive $1^{e_1}2^{e_2}3^{e_3} \leq 2^{\binom{t}{2}} \leq 2^{t^2/2}$. This implies that $(1^{e_1}2^{e_2}3^{e_3})^{n^2/t^2} \leq 2^{n^2/2}$.

To conclude the proof of part (a), note that the total number of vertex partitions is bounded above by M^n, while, for a given partition, the number of distinct multicolored cluster graphs is at most $2^{3M^2/2}$. As a consequence, we have

$$\begin{aligned} c_{3,\widehat{F}_3}(G) &\leq M^n \cdot 2^{3M^2/2} \cdot 2^{(2\eta+H(\eta))n^2} \cdot 2^{n^2/2} \\ &\leq 2^{(1+\delta)n^2/2} \end{aligned} \tag{14}$$

by our choice of n and η.

To prove part (b), assume that G is a graph with less than $\binom{n}{2} - \xi n^2$ edges and consider $\eta > 0$ such that

$$164\eta + 62H(\eta) < \xi \quad \text{and} \quad 6\eta + 2H(\eta) < 0.1. \tag{15}$$

The other constants are fixed in terms of η as in part (a), and n_1 is chosen sufficiently large so that it is larger than n_0' and n_0'' and that the last inequalities in (16) and (17) hold.

We proceed as in part (a), that is, we obtain a multicolored cluster graph R for each 3-edge-coloring of G with no rainbow triangle and its subgraph R'. Given such a graph R' on t vertices, we consider two main cases, according to whether $\overline{e}(R') = \binom{t}{2} - e(R') > (40\eta + 15H(\eta))\, t^2$, or whether this is not the case. In the former case, we have two sub-cases.

If $(40\eta + 15H(\eta))\, t^2 < \overline{e}(R') \leq t^2/4$, then Lemma 4.3 implies that

$$w(R') \leq 2^{\binom{t}{2} - 0.16\overline{e}(R')} < 2^{\binom{t}{2}(1-6\eta-2H(\eta))}.$$

If $\overline{e}(R') > t^2/4$, then Lemma 4.3 together the choice of η and t implies that

$$w(R') \leq 3^{t^2/4} = 2^{\log_2(3)t^2/4} < 2^{0.4t^2} < 2^{\binom{t}{2}(1-6\eta-2H(\eta))}.$$

In both sub-cases, we have

$$w(R') < 2^{\binom{t}{2}(1-6\eta-2H(\eta))}.$$

As in (a), summing over all possible partitions and multicolored cluster graphs, the number of good colorings of G (that yield a graph R' as in this case) is at most

$$M^n\, 2^{3M^2/2} \cdot 2^{(2\eta+H(\eta))n^2} \cdot 2^{(1-6\eta-2H(\eta))n^2/2} \leq M^n\, 2^{3M^2/2} \cdot 2^{(\frac{1}{2}-\eta)n^2} < \frac{1}{2} \cdot 2^{\binom{n}{2}}. \quad (16)$$

Next consider colorings such that $\overline{e}(R') \leq (40\eta + 15H(\eta))\, t^2$. In particular, $\overline{e}(R') < t^2/4$, so by Lemma 4.3, we have $e_3 \leq (\sqrt{2}+0.01)\overline{e} < 2\overline{e}$. Then $e_3 \leq (80\eta + 30H(\eta))\, t^2$. Once again, summing over possible partitions and cluster graphs, and using that $|E(G)| \leq \binom{n}{2} - \xi n^2$, we obtain the following upper bound on the number of such good 3-edge-colorings of G:

$$\begin{aligned} & M^n\, 2^{3M^2/2} \cdot 2^{(2\eta+H(\eta))n^2} \cdot 3^{(80\eta+30H(\eta))n^2/2} \cdot 2^{\binom{n}{2}-\xi n^2} \\ \leq\ & M^n\, 2^{3M^2/2} \cdot 2^{(164\eta+62H(\eta))n^2/2} \cdot 2^{\binom{n}{2}-\xi n^2} \leq M^n\, 2^{3M^2/2} \cdot 2^{\binom{n}{2}-\xi n^2/2} < \frac{1}{2} \cdot 2^{\binom{n}{2}} \end{aligned} \quad (17)$$

Combining equations (16) and (17), we derive $c_{3,\widehat{F}_3}(G) < 2^{\binom{n}{2}}$, which proves part (b). $\square$

We conclude this section with the following conjecture.

Conjecture 4.5. *The only extremal graph for $c_{3,\widehat{F}_3}(G)$ is the complete graph K_n.*

Comparing the number of good colorings of an almost complete graph with the number of good colorings of the complete graph seems to be hard. We did not find, for example, a way to construct an injection from the colorings of $K_n - e$ (where e is any edge) to those of K_n. Finding upper bounds for $c_{3,\widehat{F}_3}(K_n - e)$ better than those in Theorem 4.4(a) (together with better lower bound for $c_{3,\widehat{F}_3}(K_n)$) could be a first step towards the proof of the above conjecture.

Although we could not do this, our next theorem gives an upper bound for $c_{3,\widehat{F}_3}(K_n)$ which is better than what we get for taking $G = K_n$ in Theorem 4.4(a). It has a very short proof and we find that it is interesting on its own right.

Theorem 4.6. *The number of 3-edge colorings of K_n avoiding rainbow triangles satisfies*

$$c_{3,\widehat{F}_3}(K_n) \leq \frac{3}{2}(n-1)! \cdot 2^{\binom{n-1}{2}}.$$

The above theorem is an easy consequence of the following lemma.

Lemma 4.7. *Let $t \geq 2$ and consider the complete graph K_{t+1} on vertices $v_1, \ldots, v_{t+1}$. Let $\hat{K}_t$ be any 3-coloring of the edges induced by $v_1, \ldots, v_t$ which avoids a rainbow triangle. Then the number of ways to color the edges incident to v_{t+1}, still avoiding a rainbow triangle, is at most $t2^t$. In other words, $c_{3,\widehat{F}_3}(v_{t+1}, \hat{K}_t) \leq t2^t$.*

Proof. We prove this by induction on t. For $t=2$, it is easy to check that we have $c_{3,\widehat{F}_3}(v_3, \hat{K}_2) = 7 < 2 \cdot 2^2$. Assume that $t > 2$ and that the claimed result holds for smaller complete graphs. Let $v = v_{t+1}$ and fix a coloring of $\hat{K}_t$ as in the statement of this lemma. Let u be any vertex of K_t. Let N^1, N^2, and N^3 be the set of vertices in $\hat{K}_t - u$ which are adjacent to u by an edge of color 1, 2, and 3, respectively. Finally, let $n_i = |N^i|$, so that $n_1 + n_2 + n_3 = t - 1$. We count the number of ways to color the edges from v to $\hat{K}_t$ for each fixed color of the edge vu. First assume that vu receives color 1. Then all edges from v to N_2 cannot receive color 3, and all edges from v to N_3 cannot receive color 2. Therefore, there are $2^{n_2+n_3}$ ways to color the edges from v to $N_2 \cup N_3$. We argue that the number of ways to color the edges from v to N_1 is at most $n_1 2^{n_1} + 1$. In fact, this is trivial to check for $n_1 = 0$ and $n_1 = 1$. Finally, since $n_1 \leq t-1$, for $n_1 \geq 2$ we can use induction: so we can color the edges from v to N_1 in at most $n_1 2^{n_1} \leq n_1 2^{n_1} + 1$ ways. This gives a total of $(n_1 2^{n_1} + 1)2^{n_2+n_3}$ ways to color the edges from v to $N_1 \cup N_2 \cup N_3$ (given that uv is of color 1). Notice that the last bound works even when some of the sets N_i are empty. The cases in which vu is of color 2 or 3 are analogous. Adding the values in each case and using $t > 2$, gives us

$$\begin{aligned} c_{3,\widehat{F}_3}(v, \hat{K}_t) &\leq (n_1 + n_2 + n_3)2^{n_1+n_2+n_3} + 2^{n_2+n_3} + 2^{n_1+n_3} + 2^{n_1+n_2} \\ &\leq (t-1)2^{t-1} + 2^{t-1} + 2^{t-1} + 2^{t-1} \\ &= (t+2)2^{t-1} \\ &\leq t2^t. \end{aligned}$$

□

Proof of Theorem 4.6. Let $v_1, v_2, \ldots, v_n$ be any ordering of the vertices of K_n. Applying Lemma 4.7 for $t \in \{2, \ldots, n-1\}$, we obtain

$$c_{3,\widehat{F}_3}(K_n) \leq c_{3,\widehat{F}_3}(K_2) \left(\prod_{t=2}^{n-1} t2^t \right) = 3(n-1)! \cdot 2^{\binom{n-1}{2}-1}.$$

□

5. 3-colorings avoiding patterns with two colors

In this section, we prove Theorem 1.3, which we restate below.

Theorem 1.3. *Let $k \geq 3$ and let $\widehat{F}$ be a pattern of K_k with two classes, one of which induces a graph J such that $R(J,J) \leq k$. Then, for n sufficiently large, an n-vertex graph G satisfies $c_{3,\widehat{F}}(G) = c_{3,\widehat{F}}(n)$ if and only if G is isomorphic to the Turán graph $T_{k-1}(n)$.*

In particular, this works for patterns with two classes for which one of the classes is a star on at most $\lceil (k-1)/2 \rceil$ edges. For other graphs with small Ramsey number, see [18].

Our strategy to prove Theorem 1.3 is to adapt the general steps of the proof of Theorem 1.1 in Alon, Balogh, Keevash and Sudakov [1] (see also Theorem 1 in [2]) to our context. This involves proving a stability result, which shows that any graph G with a large number of good colorings is similar to $T_{k-1}(n)$, and then proving the desired result by contradiction: starting with a counterexample on n vertices, one shows that it is possible to find a counterexample on $n-1$ vertices whose 'gap' to the desired optimal solution increases. A recursive application of this step would lead to an $\sqrt{n}$-vertex graph whose number of 3-edge colorings that avoid $\widehat{F}$ is too high to be feasible.

To implement this idea, we define our concept of stability.

Definition 5.1. *A pattern $\widehat{F}$ of K_k which has at most 3 classes is said to satisfy the* 3*-stability Property if, for every $\delta > 0$, there exists n_0 as follows. If $n > n_0$ and G is an n-vertex graph such that $c_{3,\widehat{F}}(G) \geq 3^{\mathrm{ex}(n,F)}$, then there exists a partition $V(G) = V_1 \cup \cdots \cup V_{k-1}$ such that $\sum_{i=1}^{k-1} e(V_i) \leq \delta n^2$.*

Note that, if a pattern $\widehat{F}$ satisfies the 3-stability Property, then for any $\delta > 0$ and n sufficiently large, it follows immediately that $c_{3,\widehat{F}}(n) \leq 3^{\mathrm{ex}(n,F)+\delta n^2}$. To prove Theorem 1.3, we shall demonstrate two auxiliary lemmas. The first states that any pattern $\widehat{F}$ as in the statement of the theorem satisfies the 3-stability Property, while the second states that the Turán graph $T_{k-1}(n)$ is the unique extremal graph for patterns of K_k that satisfy the 3-stability Property.

Lemma 5.2. *Let $k \geq 3$ and let $\widehat{F}$ be a pattern of K_k with two classes, one of which induces a graph J such that $R(J,J) \leq k$. Then $\widehat{F}$ satisfies the* 3*-stability Property.*

Lemma 5.3. *Let $k \geq 3$ and let $\widehat{F}$ be a pattern of K_k that satisfies the* 3*-stability Property (in particular, it has at most 3 classes). For n large enough, the equality $c_{3,\widehat{F}}(G) = c_{3,\widehat{F}}(n)$ is achieved by an n-vertex graph G if and only if G is isomorphic to the Turán graph $T_{k-1}(n)$.*

As we have seen, the rainbow pattern of K_3 is a pattern that does not satisfy the 3-stability Property, so that Lemma 5.3 does not apply in this case.

5.1. Proof of Lemma 5.2

In this section, we shall prove Lemma 5.2. To this end, let $k \geq 3$ and consider a pattern $\widehat{F}$ of K_k as in the statement of the lemma. Fix $\delta > 0$, which we may assume to satisfy $\delta < 1$. Let $\beta > 0$ and m_1 be given by Erdős-Simonovits Stability (Theorem 3.5) where α is defined such that

$$\alpha \leq \frac{\delta^2}{2^9 k^2} \text{ and } \left(\frac{1-2(k-1)\sqrt{\alpha}}{1+2(k-1)\sqrt{\alpha}}\right)^{k-2} > 1 - \frac{\delta}{16}. \tag{18}$$

We may assume that $\beta \leq \alpha$.

With foresight, consider a parameter $\eta > 0$ satisfying the following inequality:

$$22H(\eta) + 44\eta < \beta. \tag{19}$$

Let $\varepsilon > 0$ and n_1 be given by the Key Lemma (Lemma 3.4), and assume that $\varepsilon < \eta/4$. Consider n_2 and M given by the Multicolored Regularity Lemma (Lemma 3.3) with $m = \max\{1/\varepsilon, m_1, (k-1)/\sqrt{\alpha}\}$. Let $n_0 \geq \max\{n_1, n_2\}$ such that (22) is satisfied for all $n \geq n_0$. In summary, our definitions lead to

$$\frac{1}{n_0} \leq \frac{1}{M} \ll \varepsilon \ll \eta \ll \beta \ll \alpha \ll \delta.$$

For $n \geq n_0$, let $G = (V,E)$ be an n-vertex graph such that $c_{3,\widehat{F}}(G) \geq 3^{\mathrm{ex}(n,F)}$ and fix an arbitrary 3-edge-coloring of G that avoids $\widehat{F}$. Consider a partition $V_1 \cup \cdots \cup V_t$ associated with this coloring given by Lemma 3.3 with $m \leq t \leq M$. Let R_1, R_2 and R_3 be the cluster graphs (with minimum density $\eta/4$) associated with each of the three colors, and let R be the corresponding multicolored cluster graph.

Exactly as in Theorem 4.4, defining E_s as the set of edges that appear in exactly s of the cluster graphs and denoting $e_s = |E_s|$, we bound the number of 3-edge colorings of G

that could give rise to this particular partition and cluster graphs (see equation (13) and the observation after it):

$$\binom{n^2}{\eta n^2} 3^{\eta n^2} (1^{e_1} 2^{e_2} 3^{e_3})^{n^2/t^2}. \tag{20}$$

To find an upper bound on $(1^{e_1}2^{e_2}3^{e_3})^{n^2/t^2}$, we define $R'_j = R_j - E_1$, so that $2e_2 + 3e_3 = e(R'_1) + e(R'_2) + e(R'_3)$. Suppose for a contradiction that $e(R'_j) > t_{k-1}(t)$, where for shortness $t_{k-1}(t)$ denotes the number $\mathrm{ex}(t, K_{k-1})$ of edges in the Turán graph $T_{k-1}(t)$. Thus there is a monochromatic copy of K_k in R'_j. For the sake of the argument, assume that it is green. Because the edges of this copy of K_k are not in E_1, it is possible to assign color red or blue to each edge e of this copy so that the edge e is an edge in the corresponding cluster graph R'_j. Since $R(J, J) \leq k$, there is a monochromatic copy of J in blue or red in this copy of K_k. Combined with green edges, we generate a copy of K_k with the forbidden pattern, which, because of Lemma 3.4, contradicts the fact that the original coloring did not contain a copy of $\widehat{F}$.

Hence $2e_2 + 3e_3 \leq 3t_{k-1}(t)$ and we obtain $\frac{e_2}{t^2} \leq \frac{3(k-2)}{4(k-1)} - \frac{3e_3}{2t^2}$. Since $2 < 3^{7/11}$, and using the bound in (11), the upper bound (20) becomes at most

$$\begin{aligned} & 2^{H(\eta)n^2} 3^{\eta n^2} (1^{e_1} 2^{e_2} 3^{e_3})^{n^2/t^2} \\ < \; & 3^{(H(\eta)+\eta)n^2 + \left(\frac{21(k-2)}{22(k-1)} + \frac{e_3}{11t^2}\right)\frac{n^2}{2}}. \end{aligned}$$

We have

$$c_{3,\widehat{F}}(G) \;\leq\; \sum_{R} 3^{(H(\eta)+\eta)n^2 + \left(\frac{21(k-2)}{22(k-1)} + \frac{e_3}{11t^2}\right)\frac{n^2}{2}}, \tag{21}$$

where the sum is over all possible partitions and their corresponding multicolored cluster graphs R defined by triples (R_1, R_2, R_3).

First assume that $e_3 < \left(\frac{k-2}{k-1} - 88\eta - 44H(\eta)\right)\frac{t^2}{2}$ for *all* such R. The number of vertex partitions is clearly bounded above by M^n, while the number of possible choices for R_1, R_2 and R_3 is at most $2^{3t^2/2} \leq 2^{3M^2/2}$. Equation (21) leads to

$$c_{3,\widehat{F}}(G) \;\leq\; M^n \cdot 2^{3M^2/2} \cdot 3^{((k-2)/2(k-1)-\eta)n^2} < 3^{t_{k-1}(n)}, \tag{22}$$

for n sufficiently large, a contradiction.

In particular there must be a multicolored cluster graph $R = (R_1, R_2, R_3)$ for which $e_3 \geq \left(\frac{k-2}{k-1} - 88\eta - 44H(\eta)\right)\frac{t^2}{2}$, where $V_1 \cup \cdots \cup V_t$ is the corresponding ε-regular partition. Let $\tilde{R} = (V(R), E_3)$ be the spanning subgraph of R with edges in E_3. By our choice of β and m, Theorem 3.5 ensures that there is a partition $W_1 \cup \cdots \cup W_{k-1}$ of the vertex set of $\tilde{R}$ with $\sum e(W_i) < \alpha t^2$.

Let $U_1 \cup \cdots \cup U_{k-1}$ be the partition of $V(G)$ given by $U_i = \cup_{j \in W_i} V_j$. We argue that the number of edges in $\cup_{i=1}^{k-1} G[U_i]$ is small. First note that:

(i) the number of edges that come from a pair of classes $(V_j, V_{j'})$ such that $\{j, j'\} \notin E_1 \cup E_2 \cup E_3$ because the pair is not ε-regular for at least one of the colors is at most $3\varepsilon t^2 (n/t)^2 = 3\varepsilon n^2$;

(ii) the number of edges that come from a pair of classes $(V_j, V_{j'})$ such that $\{j, j'\} \notin E_1 \cup E_2 \cup E_3$ because the pair is sparse for all colors is at most $3\eta n^2$;

(iii) the number of edges with both endpoints in a same set V_j is bounded above by $t(n/t)^2 = n^2/t \leq \varepsilon n^2$.

It remains to bound the number of edges in pairs $(V_j, V_{j'})$ such that $j, j' \in W_i$ for some $i \in \{1, \ldots, k-1\}$, with the additional properties that $\{j, j'\} \in E(R) = E_1 \cup E_2 \cup E_3$ and $(V_j, V_{j'})$ is ε-regular for all colors. Let $(V_j, V_{j'})$ be such a pair. In the claim below, we assume without loss of generality that $i = k-1$ to simplify the notation.

Claim 5.4. *There are no sets $V_{j_1}, \ldots, V_{j_{k-2}}$, where $j_\ell \in W_\ell$ for all $\ell \in \{1, \ldots, k-2\}$, such that both $\{j_1, \ldots, j_{k-2}, j\}$ and $\{j_1, \ldots, j_{k-2}, j'\}$ induce copies of K_{k-1} in $\tilde{R}$.*

Proof of the claim. Assume for a contradiction that there are such sets and let σ be a color for which $\{j, j'\} \in E(R_\sigma)$. This implies that $\{j_1, \ldots, j_{k-2}, j, j'\}$ induces a copy of K_k in R_σ and a copy of $K_k - \{j, j'\}$ in the cluster graphs corresponding to the other colors. This clearly leads to a copy of K_k colored according to $\widehat{F}$ (where σ is one of the colors) in the multicolored cluster graph. Lemma 3.4 leads to the desired contradiction and proves the claim.

To conclude the proof of Lemma 5.2, we find an upper bound on the number N of pairs (j, j') in a same set W_i for which there are no sets $V_{j_1}, \ldots, V_{j_{k-2}}$, one in each of the remaining classes W_ℓ, such that both $\{j_1 \ldots, j_{k-2}, j\}$ and $\{j_1, \ldots, j_{k-2}, j'\}$ induce copies of K_{k-1} in $\tilde{R}$. To this end, let B be the $(k-1)$-partite subgraph of $\tilde{R}$ induced by the classes $W_1, \ldots, W_{k-1}$, so that

$$e(B) \quad \geq \quad \left(\frac{k-2}{k-1} - 88\eta - 44H(\eta) - 2\alpha \right) \frac{t^2}{2}.$$

The following lemma implies that the size of each W_i is not far from $t/(k-1)$. A proof of this fact may be found in [12].

Lemma 5.5. *Let $H = (W, E)$ be a $(k-1)$-partite graph on t vertices with $(k-1)$-partition $W = W_1 \cup \cdots \cup W_{k-1}$. If, for some $f \geq (k-1)^2$, the graph H contains at least* $\mathrm{ex}(t, K_k) - f$ *edges, then for $i \in \{1, \ldots, k-1\}$ we have*

$$\left| |W_i| - \frac{t}{k-1} \right| \leq \sqrt{\frac{2(k-2)}{k-1} \cdot f + 2(k-2)} < \sqrt{2f}.$$

Let $\gamma = \sqrt{44\eta + 22H(\eta) + \alpha}$. Note that $\gamma \leq \sqrt{2\alpha} \leq \frac{\delta}{16k}$. Our choice of m implies that $\gamma^2 t^2 \geq \alpha t^2 \geq \alpha m^2 \geq (k-1)^2$. This allows us to apply this lemma to the graph B, and we deduce that $|W_i - t/(k-1)| \leq \gamma t$ for all $i \in \{1, \ldots, t\}$. Clearly, each of the at most $\gamma^2 t^2$ edges removed from the complete multipartite graph with classes $W_1, \ldots, W_{k-1}$ to produce B eliminates at most $(t/(k-1) + \gamma t)^{k-3}$ copies of K_{k-1}. As the number of copies of K_{k-1} in this complete multipartite graph is given by

$$|W_1||W_2| \cdots |W_{k-1}| \geq \left(\frac{t}{k-1} - \gamma t \right)^{k-1},$$

we derive that B contains at least

$$\left(\frac{t}{k-1} - \gamma t \right)^{k-1} - \gamma^2 t^2 \left(\frac{t}{k-1} + \gamma t \right)^{k-3} \tag{23}$$

copies of K_{k-1}.

Let

$$s = \frac{1}{2}\left(\frac{t}{k-1} + \gamma t\right)^{k-2}.$$

Clearly, given $i \in [k-1]$, there are at most $2s$ copies of K_{k-2} with one vertex in each set W_ℓ with $\ell \neq i$. For any edge $\{j, j'\} \in E(R)$, to avoid the occurrence of sets $V_{j_1}, \ldots, V_{j_{k-2}}$ as in Claim 5.4, no such copy K_{k-2} in $\tilde{R}$ can form copies of K_{k-1} with both j and j'. Hence at least one of j, j' lies in at most s copies of K_{k-1} in $\tilde{R}$ by the pigeonhole principle. Let A be the number of elements $j \in [t]$ which lie in at most s such copies of K_{k-1}. Clearly $N \leq (k-1) \cdot \left(\frac{t}{k-1} + \gamma t\right) \cdot A \leq 2 \cdot t \cdot A$. To find an upper bound on A, consider the auxiliary bipartite graph B' whose bipartition is given by $[t] = V(B)$ and by the set K_{k-1}^B of copies of K_{k-1} in B. We add an edge $\{u, K\}$ whenever the vertex u lies in the clique K. Clearly, A is the number of elements $j \in [t]$ with degree at most s in B'. The number of edges in B' is $(k-1)\left|K_{k-1}^B\right|$. Since every vertex $j \in [t]$ lies in at most $2s$ such copies of K_{k-1} we have

$$e(B') \leq \frac{A}{2}\left(\frac{t}{k-1} + \gamma t\right)^{k-2} + (t - A)\left(\frac{t}{k-1} + \gamma t\right)^{k-2},$$

and (23) implies that

$$e(B') \geq (k-1)\left(\frac{t}{k-1} - \gamma t\right)^{k-1} - (k-1)\gamma^2 t^2 \cdot \left(\frac{t}{k-1} + \gamma t\right)^{k-3},$$

This leads to

$$\begin{aligned}
\frac{A}{2} &\leq t - t(1-\gamma(k-1))\left(\frac{1-\gamma(k-1)}{1+\gamma(k-1)}\right)^{k-2} + \frac{(k-1)^2\gamma^2 t}{1+(k-1)\gamma} \\
&\overset{\gamma \leq \sqrt{2\alpha}}{\leq} t - t(1-\gamma(k-1))(1-\delta/16) + (k-1)\gamma t \\
&\leq 2(k-1)\gamma t + \frac{\delta}{16} t.
\end{aligned}$$

In particular, the number of edges in G with endpoints in sets $V_j, V_{j'}$ that are contained in the same W_i with the additional property that j or j' lie in at most s copies of K_{k-1} in R with one vertex in each set W_ℓ is bounded above by

$$N\left(\frac{n}{t}\right)^2 \leq 2 \cdot A \cdot \frac{n^2}{t} \leq 8(k-1)\gamma n^2 + \frac{\delta}{4} n^2.$$

Putting everything together, we conclude that the number of edges in G with endpoints in a same U_i is at most

$$3\varepsilon n^2 + 3\eta n^2 + \varepsilon n^2 + 8\gamma(k-1)n^2 + \frac{\delta}{4}n^2 \leq 4\eta n^2 + \frac{(k-1)\delta}{4k} n^2 + \frac{\delta}{4} n^2,$$

where we used $\gamma \leq \sqrt{2\alpha}$. This is less than δn^2 by our choice of $\varepsilon < \eta/4$ and $\eta \leq \beta < \delta/8$. Therefore $U_1, \ldots, U_{k-1}$ is the desired partition. This finishes the proof of Lemma 5.2. $\square$

5.2. Proof of Lemma 5.3

The aim of this section is to prove Lemma 5.3, which states that the Turán graph $T_{k-1}(n)$ is the unique $\widehat{F}$-extremal graph for any pattern $\widehat{F}$ of K_k that satisfies the 3-stability Property. We shall use the following result from [1].

Lemma 5.6 ([1])**.** *Let G be a graph and $W_1, \ldots, W_k$ be subsets of vertices of G such that, for every pair $i \neq j$ and every pair of subsets $X_i \subset W_i$ and $X_j \subset W_j$ with $|X_i| \geq 10^{-k}|W_i|$ and $|X_j| \geq 10^{-k}|W_j|$, there are at least $|X_i||X_j|/10$ edges between X_i and X_j in G. Then G contains a copy of K_k with one vertex in each set W_i.*

Now, we can give a proof for Lemma 5.3.

Proof of Lemma 5.3. Our proof is inspired by the proof of Theorem 1.1 in [1], which may be slightly shortened because of Lemma 2.9. Let $k \geq 3$ and let $\widehat{F}$ be a pattern of K_k satisfying the 3-stability Property (see Definition 5.1). For fixed $\delta > 0$, which will be chosen conveniently later, let n_0 be given as in the definition of 3-stability.

Suppose that $G = (V, E)$ is a $(3, \widehat{F})$-extremal graph on $n > n_0^2$ vertices with at least $3^{t_{k-1}(n)+m}$ distinct 3-edge colorings that avoid $\widehat{F}$, for some $m \geq 0$. We claim that, if we assume that G is not isomorphic to $T_{k-1}(n)$, then G contains a vertex v such that $G - v$ has at least $3^{t_{k-1}(n-1)+m+1}$ distinct 3-edge colorings that avoid $\widehat{F}$. Repeating this argument, we obtain a graph on n_0 vertices with at least $3^{t_{k-1}(n_0)+m+n-n_0} > 3^{n_0^2}$ such 3-colorings. This is a contradiction, as a graph on n_0 vertices has at most $n_0^2/2$ edges, and hence at most $3^{n_0^2/2}$ distinct 3-edge colorings.

To implement this idea, let G be a graph with at least $3^{t_{k-1}(n)+m}$ distinct $\widehat{F}$-free 3-edge colorings, and assume that G is not isomorphic to $T_{k-1}(n)$. The above claim holds easily if $\delta(G) < \delta_{k-1}(n)$, where $\delta_{k-1}(n)$ denotes the minimum degree of $T_{k-1}(n)$, simply choosing v as a vertex of minimum degree: in fact, each good coloring of $G - v$ can be extended to a good coloring of G in at most $3^{\delta(G)}$ ways, so we have $c(G) \leq 3^{\delta(G)}c(G - v)$, therefore $c(G - v)$ is at least

$$3^{-\delta(G)}3^{t_{k-1}(n)+m} \geq 3^{t_{k-1}(n)-\delta_{k-1}(n)+m+1} = 3^{t_{k-1}(n-1)+m+1}.$$

Thus we assume that $\delta(G) \geq \delta_{k-1}(n)$. Let $V_1 \cup \cdots \cup V_{k-1}$ be a partition of V that minimizes $\sum_i e(V_i)$, so that it satisfies $\sum_i e(V_i) < \delta n^2$, by the choice of δ and n_0 in terms of the 3-stability Property. If we fix $\delta = 10^{-11k}$, because of Lemma 5.5 we can easily claim that $||V_i| - n/(k-1)| < \sqrt{2 \cdot 10^{-11k} n^2} < \sqrt{2/10^k \cdot 10^{-10k} n^2} < \sqrt{2/10^k} \cdot 10^{-5k} n < 10^{-5k} n$.

First assume that G contains a vertex v with at least $n/(10^3 k)$ neighbors within its own class. The minimality of $\sum_i e(V_i)$ implies that v has at least this many neighbors in each of the classes V_j (otherwise we would move v to a different class). Given a good coloring of G, we say that a color σ is *rare* with respect to v and V_i if it appears at most $n/(10^{3+k})$ times in edges between v and V_i, otherwise it is called *abundant*. A class V_i is said to be *s-weak* if there are s rare colors with respect to v and V_i. More generally, a class is said to be *weak* if it is either 1- or 2-weak. Because v has a large number of neighbors in each class, note that, for every i, at least one of the colors is abundant with respect to v and V_i, and hence no class is 3-weak (recall that we only use three colors). We split the set $\mathcal{C}$ of 3-colorings of G that avoid $\widehat{F}$ into classes $\mathcal{C}_1 \cup \mathcal{C}_2$, where $\mathcal{C}_1$ contains the colorings that satisfy the following properties for some choice of colors $\sigma_1, \ldots, \sigma_{k-1}$:

(i) there is a coloring of K_k according to the pattern $\widehat{F}$ such that the $k-1$ edges incident with some vertex x have colors $\sigma_1, \ldots, \sigma_{k-1}$, respectively;

(ii) for every $i \in \{1, \ldots, k-1\}$, color σ_i is abundant with respect to v and V_i.

In other words, a coloring lies in $\mathcal{C}_1$ if it allows a partial embedding of $\widehat{F}$ into G where x is mapped to v, the neighbors of x are mapped to distinct classes V_i, and the colors between x and its neighbors are all abundant with respect to v and the respective V_i.

We observe that, if a coloring lies in $\mathcal{C}_2$, either there are three or more weak classes or there are exactly two weak classes and at least one of the classes is 2-weak. Indeed, if there are only two weak classes V_i and V_j and both are 1-weak, we may clearly choose abundant colors σ_i and σ_j with respect to V_i and V_j, respectively, regardless of whether we want them to be different or the same. Since all colors are abundant for any remaining class, we can always extend this to a partial embedding that respects the pattern $\widehat{F}$. Observe, however, that this is not necessarily the case when three classes are weak, as we can avoid monochromatic neighborhoods (by assigning distinct sets of colors to the weak classes) or a pattern of K_4 where each edge is incident with three colors (by assigning the same set of colors to all weak classes). We may also avoid monochromatic patterns if there are two weak classes, and one of them is 2-weak (as we may assign disjoint sets of colors to the weak classes).

We first consider colorings in $\mathcal{C}_1$. Let Δ be such a coloring, and let $\sigma_1, \ldots, \sigma_{k-1}$ be the colors that witness this. Consider the associated coloring of K_k according to $\widehat{F}$, where x is the vertex described in (i) and x_i denotes the neighbor of x incident with the edge of color σ_i. Let $\sigma_{i,j}$ be the color of $\{x_i, x_j\}$ in this coloring of K_k. Let $W_i \subset V_i \cap N(v)$ be the set of vertices in V_i that are adjacent to v by an edge of color σ_i, so that $|W_i| \geq n/(10^{3+k})$ for all i. We claim that the following property cannot hold:

For all distinct $j, j' \in [k-1]$, all colors σ and all $X_j \subset W_j$ and $X_{j'} \subset W_{j'}$ with $|X_j| \geq 10^{-k+1}|W_j|$ and $|X_{j'}| \geq 10^{-k+1}|W_{j'}|$, the number of edges of color σ between X_j and $X_{j'}$ is at least $|X_j||X_{j'}|/10$.

Indeed, if this property were satisfied, Lemma 5.6 would lead to a copy of K_{k-1} in G with vertex set $v_1, \ldots, v_{k-1}$ such that $v_i \in W_i$ for all i and $\{v_i, v_j\}$ has color $\sigma_{i,j}$ for all distinct i and j. The subgraph of G induced by $v, v_1, \ldots, v_{k-1}$ would be a copy of K_k colored according to $\widehat{F}$, a contradiction.

Because of this, to obtain an upper bound on $|\mathcal{C}_1|$, we may proceed as follows. There are at most 2^{2n} ways to choose X_j and $X_{j'}$ (this is a rough upper bound that uses the fact that the vertex set of the graph has 2^n possible subsets.) Moreover, once these sets are chosen, the edges between them may be colored in at most

$$\begin{aligned}
&\binom{|X_j||X_{j'}|}{|X_j||X_{j'}|/10} 2^{|X_j||X_{j'}|} 3^{|E(G)|-|E(X_j,X_{j'})|} \\
&\leq 2^{H(0.1)|X_j||X_{j'}|} 2^{|X_j||X_{j'}|} 3^{|E(G)|-|E(X_j,X_{j'})|} \leq 2^{3/2|X_j||X_{j'}|} 3^{|E(G)|-|E(X_j,X_{j'})|}
\end{aligned} \tag{24}$$

ways. By the fact that the number of edges in $E(G) - E(X_j, X_{j'})$ is at most $t_{k-1}(n) + 10^{-11k}n^2 - |X_j||X_{j'}|$, the number of colorings in $\mathcal{C}_1$ is bounded above by

$$\begin{aligned}
&2^{2n} \left(\frac{\sqrt{8}}{3}\right)^{|X_j||X_{j'}|} 3^{t_{k-1}(n)+10^{-11k}n^2} \\
&\leq \left(2^{2n} \left(\frac{\sqrt{8}}{3}\right)^{n^2/(10^{4k+4})} 3^{10^{-11k}n^2}\right) 3^{t_{k-1}(n)} \leq \left(2^{2n} 3^{(10^{-11k}-10^{-4k-6})n^2}\right) 3^{t_{k-1}(n)},
\end{aligned}$$

which is much smaller than $3^{t_{k-1}(n)}$ for sufficiently large n because $\sqrt{8}/3 \le 3^{-0.01}$.

Since $|\mathcal{C}| \ge 3^{t_{k-1}(n)+m}$ by hypothesis, this bound on $|\mathcal{C}_1|$ implies that $|\mathcal{C}_2| \ge 3^{t_{k-1}(n)+m-1}$. By our previous discussion, there are two possibilities. Firstly, there may be three weak classes V_{j_1}, V_{j_2} and V_{j_3} (this may only happen for $k \ge 4$). Secondly, there may be two weak classes V_{j_1} and V_{j_2}, where one of them, say V_{j_1}, is 2-weak.

Suppose that we are in the first case. The number of ways of choosing the classes V_{j_1}, V_{j_2} and V_{j_3} and coloring the edges between v and $V_{j_1} \cup V_{j_2} \cup V_{j_3}$ is bounded above by

$$\begin{aligned}
&(3k)^3 \binom{|V_{j_1}|}{n/10^{3k}} \binom{|V_{j_2}|}{n/10^{3k}} \binom{|V_{j_3}|}{n/10^{3k}} 2^{|V_{j_1}|+|V_{j_2}|+|V_{j_3}|} \\
&\le (3k)^3 \binom{(1/(k-1)+10^{-5k})n}{n/10^{3k}}^3 2^{(3/(k-1)+3\cdot 10^{-5k})n} \\
&\le 2^{(3\cdot H(0.001)/(k-1)+3/(k-1)+3\cdot 10^{-5k})n} \le 2^{(3.06/(k-1)+3\cdot 10^{-5k})n},
\end{aligned}$$

for large n, since $H(0.001) < 0.02$. Moreover, v is adjacent with at most $((k-4)/(k-1) + 3\cdot 10^{-5k})n$ vertices outside $V_{j_1} \cup V_{j_2} \cup V_{j_3}$, and hence the edges between v and the remainder of the graph may be colored in at most $3^{((k-4)/(k-1)+3\cdot 10^{-5k})n}$ ways. Therefore the number of ways of coloring the edges incident with v is at most,

$$2^{(3.06/(k-1)+3\cdot 10^{-5k})n} 3^{((k-4)/(k-1)+3\cdot 10^{-5k})n}.$$

In the second case, we proceed similarly. The sets V_{j_1} and V_{j_2} may be chosen in at most k^2 ways, and the edges between v and $V_1 \cup V_2$ may be colored in at most

$$\begin{aligned}
&\binom{|V_{j_1}|}{n/10^{3k}}^2 \binom{|V_{j_2}|}{n/10^{3k}} 2^{|V_{j_2}|} \le \binom{(1/(k-1)+10^{-5k})n}{n/10^{3k}}^3 2^{(1/(k-1)+10^{-5k})n} \\
&\le 2^{(3\cdot H(0.001)/(k-1)+1/(k-1)+10^{-5k})n} \le 2^{(1.06/(k-1)+3\cdot 10^{-5k})n}.
\end{aligned}$$

The remaining edges between v and the other classes may be colored in at most $3^{((k-3)/(k-1)+2\cdot 10^{-5k})n}$ ways.

If $k \ge 4$, we have at most

$$\begin{aligned}
&2^{(3.06/(k-1)+3\cdot 10^{-5k})n} 3^{((k-4)/(k-1)+3\cdot 10^{-5k})n} + 2^{(1.06/(k-1)+3\cdot 10^{-5k})n} 3^{((k-3)/(k-1)+2\cdot 10^{-5k})n} \\
\le\ & 2 \cdot 2^{(3.06/(k-1)+3\cdot 10^{-5k})n} 3^{((k-4)/(k-1)+3\cdot 10^{-5k})n},
\end{aligned}$$

ways to color the edges incident with v, and hence every good coloring of $G-v$ may be extended to at most this many distinct colorings of G. This implies that the number of colorings of $G-v$ is at least

$$\begin{aligned}
&3^{t_{k-1}(n)+m-1} \cdot 2^{-(3.06/(k-1)+4\cdot 10^{-5k})n} 3^{-((k-4)/(k-1)+3\cdot 10^{-5k})n} \\
\ge\ & 3^{t_{k-1}(n)-(k-4)/(k-1)n+m-1} 2^{-3.06n/(k-1)} 6^{-4\cdot 10^{-5k}n} \\
\ge\ & 3^{t_{k-1}(n-1)+m-1} 3^{2/(k-1)n} 2^{-3.06n/(k-1)} 6^{-4\cdot 10^{-5k}n} \\
\ge\ & 3^{t_{k-1}(n-1)+m-1} 3^{1.95/(k-1)n} 2^{-3.06n/(k-1)} \ge 3^{t_{k-1}(n-1)+m+1},
\end{aligned}$$

because $2^{3.06} < 3^{1.95}$, as required in this case.

If $k = 3$, a similar argument allows us to conclude that the number of colorings of $G-v$ is at least

$$\begin{aligned}
&3^{t_{k-1}(n)+m-1} \cdot 2^{-(1.06/(k-1)+3\cdot 10^{-5k})n} 3^{-((k-3)/(k-1)+2\cdot 10^{-5k})n} \qquad (25) \\
\ge\ & 3^{t_{k-1}(n-1)+m-1} 3^{1/(k-1)n} 2^{-1.06n/(k-1)} 6^{-3\cdot 10^{-5k}n} \\
\ge\ & 3^{t_{k-1}(n-1)+m-1} 3^{0.95/(k-1)n} 2^{-1.06n/(k-1)} \ge 3^{t_{k-1}(n-1)+m+1},
\end{aligned}$$

because $2^{1.06} < 3^{0.95}$.

Finally, we consider the case where each vertex has fewer than $n/(10^3 k)$ neighbors within their own class. First consider the case $\widehat{F} \neq T_0$ (see Figure 1). Since G is $(3, \widehat{F})$-extremal, it must be a complete multipartite graph by Theorem 1.2. Clearly, $n/(10^3 k) < (|V_i| - 2)/2$ for every $1 \leq i \leq k-1$. Assume that there is an edge vw with both ends inside some class V_i. Then, the set V_i has nonempty intersection with at least two classes of G, and hence a vertex in the smallest nonempty intersection must be adjacent to at least $|V_i|/2$ vertices in V_i, a contradiction. Hence G is a subgraph of the complete multipartite graph with classes $V_1 \cup V_2 \cup \cdots \cup V_{k-1}$, so that $e(G) \leq t_{k-1}(n)$ with equality if and only if G is isomorphic to $T_{k-1}(n)$ by Turán's Theorem.

Now, we consider the case in which $\widehat{F} = T_0$ and all vertices have fewer than $n/(3 \cdot 10^3)$ neighbors within their own class. The assumption that G has at least $3^{t_2(n)}$ distinct $\widehat{F}$-free 3-colorings and is not bipartite implies that there are vertices v and w in some class V_i, say V_1, such that vw is an edge of G.

Consider the class of colorings such that vw has color σ (for each $\sigma \in \{1,2,3\}$). Fix $u \in V_2$. If u is adjacent to both v and w, either both of the edges uv and uw are colored σ, or they both have distinct colors, none of which is σ. In particular, there are three ways to color these two edges. Of course, there are also at most three ways to color a single edge between u and the set $\{v, w\}$. Since v and w have a small number of neighbors in V_1, the number of ways of coloring the set of edges incident with v and w without producing a copy of T_0 is at most

$$3 \cdot 3^{2n/(3 \cdot 10^3)} \cdot 3^{|V_2|} \leq 3^{n/10^3} \cdot 3^{|V_2|}.$$

As in (25), and using that $|V_2| \leq n/2 + n/10^{15}$, we deduce that

$$\begin{aligned} c_{3,T_0}(G - v - w) &\geq \frac{3^{t_2(n)+m}}{3^{n/10^3 + |V_2|}} \geq 3^{t_2(n-2)+m} \cdot 3^{t_2(n) - t_2(n-2) - n/10^3 - n/2 - n/10^{15}} \\ &\geq 3^{t_2(n-2)+m+2}, \end{aligned}$$

since $t_2(n) - t_2(n-2) = n - 1$. This completes the proof. $\square$

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Appendix A. Proof of Lemma 4.2

Here we give a proof for Lemma 4.2, which holds for graphs on any number of vertices.

Proof of Lemma 4.2. Fix some natural number n. Let G be a graph on n vertices which maximizes $w(G)$, where w is given by Definition 4.1. For $v \in V(G)$ define $w(v)$ as the product of the weights of the edges incident with v. Clearly, $w(G) = \left(\prod_{v\in V(G)} w(v)\right)^{1/2}$. We will use an argument similar to the Zykov's Symmetrization proof of Turán's Theorem.

First, we show that, for any two non-adjacent vertices u, v, we must have $w(u) = w(v)$. Suppose, for a contradiction, that there are non-adjacent $u, v \in V(G)$ such that $w(v) > w(u)$. We create a new graph G^* from G by deleting u and cloning v, that is, adding a new vertex v' adjacent to the same neighbors as v. Note that, when we delete u, the weight of the remaining edges may only increase. When we add v', the weight of all edges in $G \setminus \{u\}$ will stay the same in G^*, since any edge of $G \setminus \{u\}$ belongs to a triangle in $G \setminus \{u\}$ if and only if it belongs to a triangle in G^*. Furthermore, in G^* we have $w(v') = w(v)$. Therefore, we have that $w(G^*) > w(G)$, contradicting the fact that $w(G)$ is maximum.

If all the edges of G have weight 2, then the result follows trivially. So, assume that G has an edge of weight 3, and let x be one of the endpoints of such an edge. We will prove that all vertices must have the same weight. Let $N(x)$ be the set vertices adjacent to x and $\bar{N}(x)$ be set of vertices non-adjacent to x. Moreover, for $i = 2, 3$, let $N_i(x) = \{u \in V(G) : xu \in E(G) \text{ and } w(xu) = i\}$. Note that $N_3(x)$ is non-empty (while $\bar{N}(x)$ and $N_2(x)$ may be empty). Let $y \in N_3(x)$. Notice that, since edges of weight 3 do not belong to any triangle, there can be no edges inside $N_3(x)$ or from $N_3(x)$ to $N_2(x)$ (so vertices in $N_3(x)$ are isolated in $G[N(x)]$). By the previous discussion, we have that every vertex in $\bar{N}(x)$ must have weight equal to $w(x)$ and every vertex in $N(x)$ must have weight $w(y)$. Suppose, for a contradiction, that $w(x) \neq w(y)$. If there were two non-adjacent vertices a, b with $a \in \{x\} \cup \bar{N}(x)$ and $b \in N(x)$, then we would have $w(a) = w(b)$ which implies that $w(x) = w(y)$. Therefore, the graph G must contain all edges between $\{x\} \cup \bar{N}(x)$ and $N(x)$. We claim that, in this case, $\{x\} \cup \bar{N}(x)$ and $N(x)$ must be independent sets. To prove this, let $a \in \bar{N}(x)$. Notice that, since the weight of a vertex is a number of the form $2^p 3^q$, by the unique factorization in primes, the fact that $w(a) = w(x)$ implies that $|N_i(a)| = |N_i(x)|$ for $i = 2, 3$. In particular, $|N(a)| = |N(x)|$. And since a is adjacent to all elements in $N(x)$, it follows that a cannot be adjacent to any element in $\{x\} \cup \bar{N}(x)$. Therefore, $\{x\} \cup \bar{N}(x)$ is independent. Similarly, since y is isolated in $N(x)$ and adjacent to all vertices in $\{x\} \cup \bar{N}(x)$, and for any $b \in N(x)$ we have $w(b) = w(y)$, it follows that b cannot have any neighbors in $N(x)$. Therefore, $N(x)$ is independent. It follows that G is a complete bipartite graph. But this implies that $w(G) \leq 3^{\mathrm{ex}(n,K_3)} < 2^{\binom{n}{2}} = w(K_n)$, which contradicts the fact that $w(G)$ is maximum.

Now we know that all vertices have the same weight $w(x)$. This implies that there are natural numbers $d_2, d_3, \bar{d}$ such that for all $v \in V(G)$, we have $|N_2(v)| = d_2$, $|N_3(v)| = d_3$ and $|\bar{N}(v)| = \bar{d}$. Now, as before, if it happens that $d_3 \neq 0$, we take x and y such that $y \in N_3(x)$ and note that all neighbors of y belong to $\bar{N}(x) \cup \{x\}$. This implies that $d_2 + d_3 \leq \bar{d} + 1$ and, since $d_2 + d_3 + \bar{d} = n - 1$, we have $d_2 + d_3 \leq n/2$. Therefore, $w(G) = \left(\prod_{v \in V(G)} w(v)\right)^{1/2} = \left((2^{d_2} 3^{d_3})^n\right)^{1/2} \leq 3^{n^2/4} < 2^{\binom{n}{2}}$. $\square$