# Circular Backbone Colorings: on matching and tree backbones of planar graphs☆

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**Abstract**

A (proper) $k$-coloring of a graph $G = (V, E)$ is a function $c : V(G) \to \{1, \ldots, k\}$ such that $c(u) \neq c(v)$ for every $uv \in E(G)$. Given a graph $G$ and a spanning subgraph $H$ of $G$, a circular $q$-backbone $k$-coloring of $(G, H)$ is a $k$-coloring $c$ of $G$ such that $q \leq |c(u) - c(v)| \leq k - q$ for every edge $uv \in E(H)$. The circular $q$-backbone chromatic number of $(G, H)$, denoted by $\mathrm{CBC}_q(G, H)$, is the minimum integer $k$ for which there exists a circular $q$-backbone $k$-coloring of $(G, H)$.

The Four Color Theorem implies that if $G$ is planar, we have $\mathrm{CBC}_2(G, H) \leq 8$. It is conjectured that this upper bound can be improved to 7 when $H$ is a tree, and to 6 when $H$ is a matching. In this work, we present some partial results towards these bounds.

We first prove that if $G$ is planar containing no $C_4$ as subgraph and $H$ is a linear spanning forest of $G$, then $\mathrm{CBC}_2(G, H) \leq 7$. Then, we show that if $G$ is a plane graph having no two 3-faces sharing an edge and $H$ is a matching of $G$, then $\mathrm{CBC}_2(G, H) \leq 6$. Finally, we decrease the bound and show that if $G$ is a planar graph having no $C_4$ nor $C_5$ as subgraph and $H$ is a mathing of $G$, then $\mathrm{CBC}_2(G, H) \leq 5$. Our results partially answer some questions raised by the community. In particular, the proofs use the Discharging Method, and this fact answers questions about whether one could prove such bounds for planar graphs without using the Four Color Theorem.

*Keywords:* Graph Coloring, Circular Backbone Coloring, Matching, Planar Graph, Steinberg's Conjecture.

## 1. Introduction

For basic notions and terminology on Graph Theory, the reader is referred to [1]. In this text, we only consider simple graphs.

Let $G = (V, E)$ be a graph. Given a positive integer $k$, we denote the set $\{1, \cdots, k\}$ by $[k]$. A *(proper) $k$-coloring of $G$* is a function $c : V(G) \to [k]$ such that $c(u) \neq c(v)$ for every edge $uv \in E(G)$. We say $G$ is *$k$-colorable* if there

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exists a $k$-coloring of $G$. The *chromatic number* of $G$, denoted by $\chi(G)$, is the smallest $k$ for which $G$ is $k$-colorable. We say $G$ is *$k$-chromatic* if $\chi(G) = k$. The VERTEX COLORING PROBLEM consists of determining $\chi(G)$, for a given graph $G$.

Among many practical problems that can be modeled using graph colorings, frequency assignment problems are perhaps the most famous ones [2]. There are several variations of the VERTEX COLORING PROBLEM that were defined in order to model the specific constraints of the practical applications related to frequency assignment in networks. In this context, the BACKBONE COLORING PROBLEM was defined by Broersma et al. [3, 4] to model the situation where certain channels of communication are more demanding than others.

Formally, given a graph $G$, a spanning subgraph $H$ of $G$, called the *backbone* of $G$, and two positive integers $q$ and $k$, a *$q$-backbone $k$-coloring of* $(G, H)$ is a $k$-coloring $c$ of $G$ for which $|c(u) - c(v)| \geq q$ for every $uv \in E(H)$. The *$q$-backbone chromatic number of* $(G, H)$, denoted by $\mathrm{BBC}_q(G, H)$, is the minimum $k$ for which there exists a $q$-backbone $k$-coloring of $(G, H)$. The BACKBONE COLORING PROBLEM consists of determining $\mathrm{BBC}_q(G, H)$. In this work, we focus on the case $q = 2$ and thus we usually omit $q$ from the notation.

In their seminal article, Broersma et al. observe that

$$\mathrm{BBC}(G, H) \leq 2 \cdot \chi(G) - 1. \tag{1}$$

This can be easily seen by considering a proper coloring of $G$ that uses only the $(\chi(G))$ odd colors of the set $[2 \cdot \chi(G) - 1]$. Note that, thanks to the Four Color Theorem [5, 6], whenever $G$ is a planar graph and $H$ is any spanning subgraph of $G$, we have that 7 is an upper bound for the backbone chromatic number of $(G, H)$. However, when $H$ is a spanning tree of $G$, Broersma et al. conjecture that this upper bound is actually equal to 6, and they show that this would be best possible [4].

**Conjecture 1** ([4])**.** *If $G$ is a planar graph and $T$ is a spanning tree of $G$, then*

$$\mathrm{BBC}(G, T) \leq 6.$$

In the literature, the only result approaching directly this conjecture shows that it holds whenever $T$ has diameter at most 4 [7].

The authors in [8, 9] consider a more restricted version of the Backbone Coloring Problem, where the color space is "circular", i.e., it behaves as $\mathbb{Z}/k\mathbb{Z}$, where $k$ is the total number of colors. For the remainder of this paper we are concerned only with this "circular case". In particular, whenever we add or shift colors, we do so module $k$ (where $k$ is the number of colors that are being used). We also focus only on the case where $q = 2$. More formally, given a graph $G$ and a spanning subgraph $H$ of $G$, a *circular 2-backbone $k$-coloring of* $(G, H)$ is a function $c : V(G) \to [k]$ such that, for any edge $uv$ of $H$, we have $|c(u) - c(v)| \geq 2$ (as before) and additionally $\{c(u), c(v)\} \neq \{1, k\}$. This condition is equivalent to the expression: $2 \leq |c(u) - c(v)| \leq k - 2$ for every $uv \in E(H)$. (For general $q$, the circular backbone condition is given by $q \leq |c(u) - c(v)| \leq k - q$ for every edge $uv$ of $H$). The *circular 2-backbone chromatic number* of $(G, H)$, denoted by $\mathrm{CBC}_2(G, H)$ or simply $\mathrm{CBC}(G, H)$, is the smallest $k$ for which there exists a circular 2-backbone $k$-coloring of $(G, H)$. In order to simplify the notation, we often write CBC-$k$-coloring instead of circular 2-backbone $k$-coloring.

Note that any CBC-$k$-coloring of $(G, H)$ is also a backbone $k$-coloring of $(G, H)$, and, conversely, if $c$ is a backbone $k$-coloring of $(G, H)$, then it can also be seen as a CBC-$(k+1)$-coloring of $(G, H)$. Therefore we get:

$$\mathrm{BBC}(G, H) \leq \mathrm{CBC}(G, H) \leq \mathrm{BBC}(G, H) + 1. \tag{2}$$

The following weaker circular version of Conjecture 1 is also open:

**Conjecture 2.** *If $G$ is a planar graph and $T$ is a spanning tree of $G$, then*

$$\mathrm{CBC}(G, T) \leq 7.$$

One may observe that a graph $G$ whose chromatic number is $k$, satisfies $\mathrm{CBC}(G, H) \leq 2k$, by combining inequalities (1) and (2). In general, this is best possible as, for $G = H$, one can easily check that $\mathrm{CBC}(G, G) = 2\chi(G)$.

Steinberg conjectured that every planar graph $G$ having no $C_4$ or $C_5$ as subgraph satisfies $\chi(G) \leq 3$ [10]. This famous conjecture was very recently disproved by Cohen-Addad, Hebdige, Kral, Li, and Salgado [11]. However, we may still hope that the following weaker version of the conjecture is true.

**Conjecture 3.** *If $G$ is a planar graph having no $C_4$ or $C_5$ as subgraph and $H$ is a spaning tree of $G$, then* $\mathrm{CBC}(G, H) \leq 6$.

The case where the backbone graph is a matching has also been considered in the literature. It has been proved, using the Four Color Theorem, that $\mathrm{BBC}(G, M) \leq 6$ whenever $G$ is a planar graph and $M$ is a matching in $G$, and that this upper bound cannot be improved to 4 [12]. Then, the following questions are posed:

**Question 1.** *Let $G$ be a planar graph and $M$ be a matching in $G$. Is it possible to prove* $\mathrm{BBC}(G, M) \leq 6$ *without using the Four Color Theorem?*

**Question 2.** *Let $G$ be a planar graph and $M$ be a matching in $G$. Does* $\mathrm{BBC}(G, M) \leq 5$ *hold?*

In this paper, we prove particular cases of Conjectures 2 and 3, and give a partial answer to Question 1 and to the circular version of Question 2. We describe our results in the following subsections.

### *1.1. Matching Backbones*

Recall that in the definition of backbone colorings, the backbone $H$ of a graph $G$ is a spanning subgraph of $G$. By definition, a matching $M$ is only a set of independent edges of $G$. With a slight abuse of notation, whenever we refer to a matching $M$ as backbone of a graph $G$, we mean the subgraph of $G$ whose vertex set is equal to $V(G)$ and whose edge set is $M$. In fact, any subgraph $H$ of $G$ may be thought as the backbone $\tilde{H} = (V(G), E(H))$, as adding isolated vertices to $H$ does not impose any extra conditions on the colorings.

It is known that if $G$ is a 3-colorable graph and $M$ is a matching of $G$, then $\mathrm{BBC}(G, M) \leq 4$ [12]. Combining this result with inequality (2), we observe that if Steinberg's Conjecture were true, it would trivially imply that $\mathrm{CBC}(G, M) \leq 5$, whenever $G$ is a planar graph without cycles of length 4 or 5, and $M$ is a matching of $G$. We prove that this bound holds, regardless of Steinberg's Conjecture being false. This proves a weaker version of Conjecture 3 and gives a partial answer to Question 2.

**Theorem 4.** *If $G$ is a planar graph without cycles of length 4 or 5 as subgraph, and $M$ is a matching of $G$, then* $\mathrm{CBC}(G, M) \leq 5$.

The following theorem partially answers Question 1 and the circular version of Question 2. Recall that a plane $G$ is a planar graph $G$ together with a given embedding of $G$ in the plane. We say that two faces of a plane graph $G$ are *adjacent* if they have an edge in common.

**Theorem 5.** *If $G$ is a (simple) plane graph with no adjacent 3-faces and $M$ is a matching of $G$, then* $\mathrm{CBC}(G, M) \leq 6$.

Although our result restricts the class of graphs when compared to the result presented in [12], it is stronger on this restricted class since we deal with *circular* backbone colorings instead. We emphasize that our result also points to a positive answer to the question about whether $\mathrm{BBC}(G, M) \leq 5$, and that our proof does not use the Four Color Theorem.

### *1.2. Linear Forest Backbones*

Our main result concerns a more general backbone. A forest is called *linear* if each of its components are paths.

In [13], Araújo et al. investigate $\mathrm{CBC}(G, F)$ when $F$ is a forest. They prove that if $G$ is a planar graph with no cycles of length 4 or 5, then $\mathrm{CBC}(G, F) \leq 7$ whenever $F$ is a spanning forest of $G$, and that $\mathrm{CBC}(G, F) \leq 6$, whenever $F$ is a spanning linear forest of $G$ [13]. Observe that their results partially solve Conjectures 2 and 3.

Our last result is similar to theirs in nature, but we consider only the case where the backbone $H$ is a linear forest.

**Theorem 6.** *If $G$ is a planar graph without cycles of length 4 as subgraph, and $F$ is a (spanning) linear forest of $G$, then* $\mathrm{CBC}(G, F) \leq 7$.

Note that, in the Theorem 6, $G$ is allowed to have a $C_5$ as subgraph, but we are allowed to use an extra color compared to the result in [13]. However, this was expected since our efforts were done towards an answer to Conjecture 2.

The remainder of this text is organized as follows: in Section 2, we introduce some notation and state known results. In Sections 3, 4 and 5, we prove Theorems 4, 5 and 6, respectively.

## 2. Preliminaries

We prove all theorems of this paper (Theorems 4, 5 and 6) by contradiction, considering the existence of a minimal counterexample and using the Discharging Method. Let us first properly define what minimal means in this context.

In all proofs, we consider the following partial order defined over the set of pairs of graphs $(G, H)$ such that $H$ is a spanning subgraph $G$. For such a pair, we write $(G', H') \preceq (G, H)$, whenever $G' \subseteq G$, $H' \subseteq H$ and $H'$ is a spanning subgraph of $G'$. When $(G', H') \preceq (G, H)$, we call $(G', H')$ a *subpair* of $(G, H)$. We say that $(G', H')$ is a *proper* subpair of $(G, H)$ if it is a subpair of $(G, H)$ such that $G' \subset G$ or $H' \subset H$ (or both).

All our theorems state that $\mathrm{CBC}(G, H) \leq k$, for pairs $(G, H)$ satisfying a given condition and a particular positive integer $k$. We say that a pair $(G, H)$ is

$k$-minimal when $\text{CBC}(G, H) > k$ and $\text{CBC}(G', H') \leq k$ for every proper subpair $(G', H')$ of $(G, H)$.

A $k$-minimal counterexample to one of our theorems is simply a $k$-minimal pair which is also a counterexample to the theorem. Since the hypothesis in each of Theorems 4, 5 and 6 are monotone, whenever there is a counterexample to the statement $\text{CBC}(G, H) \leq k$, there is also a $k$-minimal counterexample. We may omit the number $k$ when it is clear in the context.

In most proofs of ours auxiliary results, given a $k$-minimal counterexample $(G, H)$, we get a contradiction by extending a CBC-$k$-coloring of a subpair $(G', H')$ to a CBC-$k$-coloring of $(G, H)$.

We shall use the following lemma that was presented in [13]. Since the proof is short and illustrates the idea above, we also show it here. Notice that in this lemma, the pair $(G, H)$ does not need to be a counterexample to any of the theorems.

**Lemma 7** (Lemma 16 in [13])**.** *If $(G, H)$ is $k$-minimal, then for every $u \in V(G)$, we have that $d_G(u) + 2d_H(u) \geq k$.*

*Proof.* Assume, for a contradiction, that there is a vertex $u$ of $G$ such that $d_G(u) + 2d_H(u) < k$. Consider a CBC-$k$-coloring of $(G - u, H - u)$, which exists by the minimality of $(G, H)$. One can see that trying to extend this coloring to a CBC-$k$-coloring of $G$, at most $d_G(u) + 2d_H(u)$ colors are forbidden for $u$. As $d_G(u) + 2d_H(u) < k$, there is a color available for $u$ and the extension is feasible. Therefore, $(G, H)$ admits a CBC-$k$-coloring, and this contradicts the fact that $(G, H)$ is $k$-minimal. □

As we shall use the above argument many times, it is convenient to introduce the following notation.

If $\psi$ is a coloring of a graph $G$ and $S \subseteq V(G)$, then denote by $\psi(S)$ the set $\{\psi(u) : u \in S\}$. Given $c \in [k]$, we denote by $\langle c \rangle$ the set $\{d \in [k] : |c - d| \leq 1 \text{ or } |c - d| \geq k - 1\}$. This is equivalent to work with colors in $\mathbb{Z}_k = \mathbb{Z}/k\mathbb{Z}$ and set $\langle c \rangle = \{c - 1, c, c + 1\}$ module $k$. Finally, we denote the power set of $[k]$ by $2^{[k]}$.

Given a pair $(G, H)$ and a subgraph $G' \subset G$, consider the backbone $H'$ of $G'$ such that $H' = (V(G'), E(G') \cap E(H))$. For a CBC-$k$-coloring $\psi$ of the subpair $(G', H')$ and a vertex $u \in V(G) \setminus V(G')$, we define the set of *available colors for $u$ in $\psi$* as the set of colors that can be used in $u$ to extend $\psi$ to a CBC-$k$-coloring of the pair $(G' \cup \{u\}, H' \cup \{u\})$. (Here the subgraph $G' \cup \{u\}$ includes all edges of $G$ from $u$ to $V(G')$ and $H' \cup \{u\}$ includes all edges of $H$ from $u$ to $V(G')$). More precisely, we set:

$$A_\psi(u) = [k] \setminus \Big(\psi\big(N_G(u) \cap V(G')\big) \cup \big(\bigcup \big\{\langle\psi(x)\rangle : x \in N_H(u) \cap V(G')\big\}\big)\Big).$$

Also, we denote $|A_\psi(u)|$ by $a_\psi(u)$. We will also write only $A_\psi$ to represent the function $A_\psi : V(G) \setminus V(G') \to 2^{[k]}$ that we have just defined.

In Sections 3 and 4, where the backbone is a matching, we will use the following straight consequence of Lemma 7.

**Proposition 8.** *Let $(G, M)$ be a $k$-minimal pair in which $k \geq 4$ and $M$ is a matching. The following statements hold.*

1. *$\delta(G) \geq k - 2$.*

2. *If* $d_G(u) = k-2$*, then there is* $w \in V(G)$ *such that* $uw \in M$ *and* $d_G(w) \geq k$*.*

*Proof.* By Lemma 7, since $d_M(u) \leq 1$, we get that $d_G(u) \geq k-2$ for every $u \in V(G)$. This immediately proves statement 1. By the same argument, if $d_G(u) \leq k-1$, we must have $d_M(u) = 1$. In particular, assuming that $d_G(u) = k-2$, there must be some $w \in V(G)$ such that $uw \in M$. By contradiction, suppose that $d_G(w) \leq k-1$. Let $\psi$ be a CBC-$k$-coloring of $(G-u-w, M-u-w)$ (which exists by the minimality of $(G, M)$). Note that $|N_G(u) \cap V(G-u-v)| = k-3$ and $|N_M(u) \cap V(G-u-v)| = 0$, therefore $a_\psi(u) \geq 3$. Similarly, $a_\psi(w) \geq 2$. From this, it is easy to conclude that there exists a color $c \in A_\psi(w)$ such that $A_\psi(u) \setminus \langle c \rangle \neq \emptyset$. This implies that $\psi$ can be extended to a CBC-$k$-coloring of $(G, M)$, a contradiction. □

Finally, if $G$ is a plane graph, then denote by $F(G)$ the set of faces of $G$. If a face $f \in F(G)$ has degree $i$, we say that $f$ is an $i$-face and that $|f| = i$. Denote by $F_i(G)$ the set of faces of degree $i$ and set $f_i(G) = |F_i(G)|$. Moreover, if $u \in V(G)$, then we say that $F_i^G(u)$ is the set of faces of degree $i$ containing $u$ in $G$. Similarly, we denote $|F_i^G(u)|$ by $f_i^G(u)$. We often omit $G$ from these notations when it is clear from the context.

## 3. Matching backbones on planar graphs with no $C_4$ nor $C_5$

The goal of this section is to prove Theorem 4, which is restated below:

**Theorem 4.** *If* $G$ *is a planar graph without cycles of length* 4 *or* 5 *as subgraph, and* $M$ *is a matching of* $G$*, then* $\mathrm{CBC}(G, M) \leq 5$*.*

In order to do so, the only extra information we need is given by the following result from [13].

**Lemma 9** ([13])**.** *Let* $G$ *be a plane graph without cycles of length* 4 *or* 5 *such that* $G \neq K_3$*. Then,*

$$\sum_{v \in V(G)} (d_G(v) - 3) \leq \frac{3f_3(G)}{2} - 6. \tag{3}$$

Now, we may proceed to the proof of the main result of this section.

*Proof of Theorem 4.* Let $(\widehat{G}, \widehat{M})$ be a 5-minimal counterexample to Theorem 4, i.e., $\widehat{G}$ is a plane graph having no $C_4$ or $C_5$ as subgraph, $\widehat{M}$ is a matching of $G$, $\mathrm{CBC}(\widehat{G}, \widehat{M}) \geq 6$ and $\mathrm{CBC}(G', M') \leq 5$ for every proper subpair $(G', M')$ of $(\widehat{G}, \widehat{M})$. Clearly, $\widehat{G} \neq K_3$.

We shall use the Discharging Method to show that $\widehat{G}$ does not satisfies equation (3), contradicting the statement of Lemma 9. Therefore, such a minimal counterexample cannot exist. By Proposition 8 (with $k = 5$), we already have a simple structural result about $\widehat{G}$. We have that

1. $\delta(\widehat{G}) \geq 3$, and
2. if $d_{\widehat{G}}(u) = 3$, then there is $u^* \in V(\widehat{G})$ such that $uu^* \in \widehat{M}$ and $d_{\widehat{G}}(u^*) \geq 5$.

For the remainder, we will use only $d(u)$ to denote $d_{\widehat{G}}(u)$.

We now proceed to the Discharging Method. Take any embedding of $\widehat{G}$ in the plane. We attribute charges to every vertex, and every face of this embedding. Give charge $d_{\widehat{G}}(v) - 3$ for every vertex $v \in V(\widehat{G})$, charge $-\frac{3}{2}$ for every face $f \in F_3(\widehat{G})$, and charge 0 to every other face. In the sequel, we redistribute these charges between vertices and faces of $\widehat{G}$ in such a way that, at the end, each vertex and each face has nonnegative charge while the total sum of the charges does not change. Because of this, we conclude that

$$\sum_{v \in V(\widehat{G})} (d_{\widehat{G}}(v) - 3) - \frac{3f_3(\widehat{G})}{2} \geq 0,$$

and this contradicts Lemma 9.

In order to redistribute the charges, we apply the following discharging rules:

**Rule 1.** *For each $u \in V(\widehat{G})$ such that $d(u) = 3$, send charge $\frac{1}{2}$ from $u^*$ to $u$.*

**Rule 2.** *For each vertex $u \in V(\widehat{G})$ and each face $f \in F_3(u)$, send charge $\frac{1}{2}$ from $u$ to $f$.*

We emphasize that our discharging procedure first applies Rule 1 (simultaneously) to all vertices $u$, and only afterwards it applies Rule 2 (simultaneously) to all faces of $F_3(G)$. Note that the charge of every face not in $F_3(G)$ remains zero.

For each $x \in V(\widehat{G}) \cup F_3(\widehat{G})$, denote by $\mu_0(x)$, $\mu_1(x)$, $\mu_2(x)$ the charge of $x$ before Rule 1 has been applied, before Rule 2 has been applied, and after Rule 2 has been applied, respectively. Recall that $\mu_0(u) = d(u) - 3$ for every $u \in V(\widehat{G})$, and $\mu_0(f) = -\frac{3}{2}$ for every $f \in F_3(\widehat{G})$. Because $\widehat{M}$ is a matching, no vertex sends charge to more than one other vertex, and the condition on the degree of the vertices $u^*$ imply that:

- If $d(u) = 3$, then $\mu_1(u) = \frac{1}{2}$;
- If $d(u) = 4$, then $\mu_1(u) = \mu_0(u) = 1$; and
- If $d(u) \geq 5$, then

$$\mu_1(u) \geq \mu_0(u) - \frac{1}{2} = \frac{2d(u) - 7}{2}.$$

Moreover, no face changes its charge by Rule 1, so $\mu_1(f) = \mu_0(f)$ for every face $f \in F_3(\widehat{G})$. Let $u$ be any vertex. Note that, since $\widehat{G}$ has no cycles of length 4, no two faces in $F_3(\widehat{G})$ can share an edge. This implies that $f_3(u) \leq \lfloor \frac{d(u)}{2} \rfloor$. One can verify, for each of the previous cases on $d(u)$, that $\mu_1(u) \geq \frac{1}{2} \lfloor \frac{d(u)}{2} \rfloor$. Therefore, $\mu_1(u) \geq \frac{f_3(u)}{2}$. This means that after sending charge 1/2 to each $f \in F_3(u)$, the vertex $u$ still has non-negative charge, i.e., $\mu_2(u) \geq 0$. On the other hand, each $f \in F_3(\widehat{G})$ receives 1/2 units of charge from each vertex in $f$, thus we have that $\mu_2(f) = \mu_1(f) + 3/2 = \mu_0(f) + 3/2 = 0$. This finishes the proof of Theorem 4. □

## 4. Matching backbones on plane graphs with no adjacent 3-faces

In this section, we consider a plane graph $G$. We still have a matching $M$ as backbone, however we weaken the constraints on the structure of $G$. On the other hand, we need an extra color compared to the main result in the previous section. Let us restate our result here.

**Theorem 5.** *If $G$ is a plane graph with no adjacent 3-faces and $M$ is a matching of $G$, then* $\mathrm{CBC}(G, M) \leq 6$.

In order to prove Theorem 5, we need to study in more details the structure of a hypothetical minimal counterexample for it. Consequently, we need to introduce some extra notation.

Let $G$ be a plane graph and $G^*$ be its dual. Recall that $F_3(G)$ is the set of faces of degree 3 in $G$, so it is as well the set of vertices of degree 3 in $G^*$. We denote the graph $G^* - F_3$ by $G_4^*$, i.e., the subgraph of $G^*$ induced by faces of degree (in $G^*$) at least 4 . We say that a component of $G_4^*$ is an *island* of $G$. In other words, an island of $G$ is a maximal set of faces of $G$ that have degree at least four and form a connected component in the dual. Moreover, if $H$ is an acyclic component of $G_4^*$ such that $d_{G^*}(f) = 4$ for every $f \in V(H)$, then we say that $H$ is a *bad island* of $G$. We denote the set of bad islands of $G$ by $\Gamma(G)$ and we let $\gamma(G)$ denote $|\Gamma(G)|$.

Let $f \in F_3(G)$ and $H$ be an island of $G$. We say that $f$ is *adjacent to $H$*, or $f$ shares an edge with $H$, if there exists a face $f' \in V(H)$ such that $f$ and $f'$ are adjacent. Also, we denote by $\Gamma(G, f)$ the set of bad islands that are adjacent to $f$ in the plane graph $G$.

**Lemma 10.** *Let $G$ be a plane graph with no two adjacent 3-faces. Then*

$$3f_3(G) + f_4(G) \leq |E(G)| + \gamma(G).$$

*Proof.* Let $E_3 = \{e \in E(G) : e \text{ is in the boundary of some face of degree 3}\}$ and $\overline{E}_3 = E(G) \setminus E_3$. Observe that $|E_3| = 3f_3(G)$, because $G$ has no two faces of degree 3 sharing an edge. In the sequel, we prove that $|\overline{E}_3| \geq f_4(G) - \gamma(G)$. This implies Lemma 10, as $|E(G)| = |E_3| + |\overline{E}_3|$.

Note that if $e \in \overline{E}_3$, then the edge $e^*$ corresponding to $e$ in the dual $G^*$ belongs to $G_4^*$. Conversely, any edge $e^* \in E(G_4^*)$, corresponds to an edge $e \in E(G)$ that does not belong to the boundary of any face in $F_3(G)$. Hence, $e \in \overline{E}_3$. Therefore, $|\overline{E}_3| = |E(G_4^*)|$. Let $\mathrm{i}(G_4^*)$ be the number of acyclic components of $G_4^*$ (by definition, this is the number of islands of $G$). Finally, because the number of edges in any graph is at least its number of vertices minus the number of acyclic components of the graph, we get:

$$|\overline{E}_3| = |E(G_4^*)| \geq |V(G_4^*)| - \mathrm{i}(G_4^*).$$

Also, note that

$$|V(G_4^*)| - f_4(G) \geq \mathrm{i}(G_4^*) - \gamma(G),$$

as in every island that is not a bad island we must have at least one vertex of $G_4^*(G)$ that has degree at least 5 (and therefore has not been counted on $f_4(G)$). Rearranging the terms we get $|\overline{E}_3| \geq f_4(G) - \gamma(G)$ as we wanted. □

**Lemma 11.** *Let $G$ be a plane graph with no two adjacent 3-faces. Then,*

$$\sum_{v\in V(G)} (d_G(v)-5) + f_3(G) - \gamma(G) \le -10.$$

*Proof.* We compute $\sum_{f\in F(G)}(|f|-5)$ in two different ways. First notice that

$$\begin{aligned}\sum_{f\in F(G)} (|f|-5) = \sum_{k=3}^{\infty}(k-5)f_k(G) &\ge -2f_3(G) - f_4(G)\\ &\ge -|E(G)| - \gamma(G) + f_3(G), \end{aligned} \tag{4}$$

where we used Lemma 10 to obtain the last inequality.

On the other hand,

$$\sum_{f\in F(G)} (|f|-5) = \sum_{f\in F(G)} (|f|) - 5|F(G)| = 2|E(G)| - 5|F(G)|. \tag{5}$$

Combining inequalities (4) and (5) with Euler's Formula, we obtain:

$$2|E(G)| - 5(2 - |V(G)| + |E(G)|) \ge -|E(G)| - \gamma(G) + f_3(G),$$

and therefore

$$2|E(G)| - 5|V(G)| - \gamma(G) + f_3(G) \le -10.$$

This clearly implies the Lemma 11. □

Next, we use the Discharging Method to argue that the existence of a counterexample to Theorem 5 contradicts Lemma 11.

*Proof of of Theorem 5.* By contradiction, let $(\widehat{G}, \widehat{M})$ be a 6-minimal counterexample to Theorem 5. This means that $\widehat{G}$ is a plane graph having no two adjacent 3-faces, $\widehat{M}$ is a matching of $\widehat{G}$, $\mathrm{CBC}(\widehat{G}, \widehat{M}) \ge 7$ and $\mathrm{CBC}(G', M') \le 6$ for every proper subpair $(G', M')$ of $(\widehat{G}, \widehat{M})$.

As in the previous section, Proposition 8 (now with $k = 6$) implies the following immediate structural result for $(\widehat{G}, \widehat{M})$.

1. $\delta(\widehat{G}) \ge 4$.
2. If $d_{\widehat{G}}(u) = 4$, then there is $u^* \in V(\widehat{G})$ such that $uu^* \in \widehat{M}$ and $d_{\widehat{G}}(u^*) \ge 6$.

We now assign charges to vertices, faces and islands of $\widehat{G}$. We assign charge $d_{\widehat{G}}(v) - 5$ to each vertex $v \in V(\widehat{G})$, charge 1 to each face $f \in F_3(\widehat{G})$, and charge $-1$ to each bad island $b \in \Gamma(\widehat{G})$ (and zero to everything else). Our goal is to prove that the sum of all these charges, that is,

$$\sum_{v\in V(\widehat{G})} (d_{\widehat{G}}(v)-5) + f_3(\widehat{G}) - \gamma(\widehat{G}),$$

is non-negative. Observe that this is a direct contradiction to Lemma 11.

In order to achieve this goal, we apply the following discharging rules to redistribute the charges (without changing their total sum). We claim that after applying such rules, every vertex, face and island of $\widehat{G}$ ends with a non-negative charge.

**Rule 1.** *For each $f \in F_3(\widehat{G})$, send charge $\frac{1}{3}$ from $f$ to each $b \in \Gamma(\widehat{G}, f)$.*

**Rule 2.** *For each $u \in V(\widehat{G})$ such that $d_{\widehat{G}}(u) = 4$, send charge 1 from $u^*$ to $u$.*

In this particular proof, the rules could be applied in any order since the first rule only discharges from faces to islands, while the second only does between vertices. But in order to analyze their effects, we will assume that we first apply Rule 1 as far as possible and later we apply Rule 2 as far as possible. Clearly, any face that is not in $F_3(\widehat{G})$ and any island that is not bad ends up with non-negative charge.

For each $x \in V(\widehat{G}) \cup F_3(\widehat{G}) \cup \Gamma(\widehat{G})$, let $\mu_0(x), \mu_1(x), \mu_2(x)$ denote the initial charge of $x$, the charge of $x$ after Rule 1 is applied to all faces, and the charge of $x$ after Rule 2 is applied to all vertices, respectively. Recall that $\mu_0(v) = d(v) - 5$ for every $v \in V(\widehat{G})$; $\mu_0(f) = 1$ for every $f \in F_3(\widehat{G})$; and $\mu_0(b) = -1$ for every $b \in \Gamma(\widehat{G})$.

First observe that, since $\widehat{M}$ is a matching and by the conditions on $\delta(\widehat{G})$ and $d_{\widehat{G}}(u^*)$, we get that $\mu_1(v) \geq 0$ for every $v \in V(\widehat{G})$.

It remains only to prove that each bad island also ends with a non-negative charge. So, consider a bad island $H$ of $\widehat{G}$, i.e., $H$ is an acyclic component of $\widehat{G}^*_4$ such that such that $d_{\widehat{G}^*}(f) = 4$ for every $f \in V(H)$. Assume first that $V(H)$ is a singleton, say $V(H) = \{f\}$. Each face of degree 3 in $\widehat{G}$ shares at most two edges with $f$. However, because two faces of degree 3 in $\widehat{G}$ intersect each other in at most one vertex, we get that $f$ is adjacent to at least three faces of $F_3(\widehat{G})$. If $|V(H)| \geq 2$, then $H$ has at least two leafs; as before, we get that $H$ is adjacent to at least three distinct faces of $F_3(\widehat{G})$. In any case, we get that $y = |\{f \in F_3(\widehat{G}) : H \in \Gamma(\widehat{G}, f)\}| \geq 3$, which implies that $\mu_2(H) = \mu_1(H) = \mu_0(H) + y/3 \geq 0$. □

## 5. Linear Forest Backbone

In this section, we prove Theorem 6. Let us recall its statement.

**Theorem 6.** *If $G$ is a planar graph without cycles of length 4 as subgraph, and $F$ is a linear spanning forest of $G$, then* $\mathrm{CBC}(G, F) \leq 7$.

We use the same general strategy, but the structural properties needed here are more complex. In the previous sections, the very simple Proposition 8, whose proofs considered the removal of only two vertices of a minimal counterexample, was enough to guarantee a structure in which we could apply the Discharging Method. Here, the backbone $H$ is a linear forest and we shall need to remove entire subpaths from a minimal counterexample $(G, H)$. To extend a coloring $\psi$ of a subpair of $(G, H)$ to a coloring of $(G, H)$, we shall need to work with the lists $A_\psi$ in a more clever way. Let us start by proving some tool lemmas in the next subsection.

### *5.1. Forbidden substructures in a minimal counterexample*

In order to do the extension of a coloring of a subpair to a coloring of the pair, it is convenient to work with list colorings instead of usual colorings. Let $(G, H)$ be such that $H$ is a backbone of $G$, $k$ be a positive integer, and $\mathcal{L} : V(G) \to 2^{[k]}$ be a function that associates to each vertex a list of colors.

If there exists a CBC-$k$-coloring $\psi$ of $(G, H)$ such that $\psi(v) \in \mathcal{L}(v)$, for all $v \in V(G)$, then we say that $\psi$ is an $\mathcal{L}$-CBC-$k$-coloring of $(G, H)$ and that $(G, H)$ is *$\mathcal{L}$-CBC-$k$-colorable.* Throughout the proof, we often consider a worst-case scenario and suppose $|\mathcal{L}(v)|$ to be as small as possible in the context for every vertex $v \in V(G)$. This is not a problem since whenever $(G, H)$ is $\mathcal{L}$-CBC-$k$-colorable and $\mathcal{L}'$ is such that $\mathcal{L}(v) \subseteq \mathcal{L}'(v)$ for every $v \in V(G)$, we also have that $(G, H)$ is $\mathcal{L}'$-CBC-$k$-colorable.

We use the reduction rule below to eventually prove that certain substructures are forbidden in a 7-minimal counterexample to Theorem 6.

Consider a pair $(H, P)$ such that $|V(H)| \geq 2$ and $P$ is a Hamiltonian path of $H$. Let us write $P$ as $(v_1, \ldots, v_n)$. Also, let $\mathcal{L} : V(H) \to 2^{[7]}$ be a list assignment for $H$, and $\mathcal{L}' : V(H') \to 2^{[7]}$ be a list assignment for some $H' \subseteq H$. We denote the values $|\mathcal{L}(x)|$ and $|\mathcal{L}'(x)|$ by $\ell(x)$ and $\ell'(x)$, respectively.

**Reduction Rule**: given that $P$ is a Hamiltonian path of $H$, we say that $((H', P'), \mathcal{L}')$ is a *reduction of* $((H, P), \mathcal{L})$ *on* $v_1$ if all the following conditions hold:

(i) $H' = H - v_1$,

(ii) $P' = P - v_1$,

(iii) $\ell'(v_2) \geq \ell(v_2) - 2$,

(iv) $\ell'(x) \geq \ell(x) - 1$ for every $x \in N_H(v_1) \setminus \{v_2\}$,

(v) $\ell'(x) = \ell(x)$ for every $x \in V(H') \setminus N_H(v_1)$, and

(vi) If $\mathcal{L}(v_2) \setminus \mathcal{L}'(v_2) = \{c, d\}$, then $|\langle c \rangle \cup \langle d \rangle| \leq 5$.

We say that a reduction $((H', P'), \mathcal{L}')$ of $((H, P), \mathcal{L})$ on $v_1$ is *reversible* if every $\mathcal{L}'$-CBC-$k$-coloring of $(H', P')$ can be extended to an $\mathcal{L}$-CBC-$k$-coloring of $(H, P)$.

The following lemma gives a sufficient condition depending only on the structure of the list and the degree of the removed vertex $(v_1)$ for $((H, P), \mathcal{L})$ to have an reversible reduction.

**Lemma 12.** *Let $H$ be any graph, $P = (v_1, \ldots, v_n)$ be a Hamiltonian path of $H$, and consider $\mathcal{L} : V(H) \to 2^{[7]}$. If the conditions below hold, then $((H, P), \mathcal{L})$ has an reversible reduction on $v_1$.*

1. *$d_H(v_1) \leq 4$;*
2. *$\ell(v_1) \geq 1 + d_H(v_1)$; and*
3. *If $d_H(v_1) = 4$, and $c$ and $d$ are the colors not in $\mathcal{L}(v_1)$, then $|\langle c \rangle \cup \langle d \rangle| \leq 5$.*

*Proof.* In each case of this proof, we present an reversible reduction $((H', P'), \mathcal{L}')$ of $((H, P), \mathcal{L})$ on $v_1$. In all cases, we have $H' = H - v_1$ and $P' = P - v_1$. Thus, we just need to present the list assignment $\mathcal{L}'$ in each step satisfying Statements (iii)-(vi) and in such a way that any $\mathcal{L}'$-CBC-7-coloring of $(H', P')$ can be extended to a $\mathcal{L}$-CBC-7-coloring of $(H, P)$.

Without loss of generality, suppose that $\ell(v_1) = 1 + d_H(v_1)$. First, suppose that $d_H(v_1) = 1$. If $\mathcal{L}(v_1)$ has two consecutive colors, say $c$ and $c + 1$, for some

$c \in [7]$, then set $\mathcal{L}'(v_2) = \mathcal{L}(v_2) \setminus \{c, c+1\}$. If $\mathcal{L}(v_1) = \{c-1, c+1\}$ for some $c \in [7]$, then set $\mathcal{L}'(v_2) = \mathcal{L}(v_2) \setminus \{c\}$. Otherwise, let $\mathcal{L}'(v_2) = \mathcal{L}(v_2)$. Finally, let $\mathcal{L}'(x) = \mathcal{L}(x)$ for every $x \in V(H) \setminus \{v_1, v_2\}$. One can see that $((H - v_1, P - v_1), \mathcal{L}')$ is a reduction of $((H, P), \mathcal{L})$ on $v_1$. Let us argue that any $\mathcal{L}'$-CBC-7-coloring of $(H - v_1, P - v_1)$ can be extended to an $\mathcal{L}$-CBC-7-coloring of $(H, P)$. If no such coloring exists, then the lemma holds by vacuity. Otherwise, let $\psi$ be an $\mathcal{L}'$-CBC-7-coloring of $(H - v_1, P - v_1)$. By the choice of $\mathcal{L}'$ in each case, note that $\mathcal{L}(v_1) \setminus \langle \psi(v_2) \rangle \neq \emptyset$, and as $v_2$ is the only neighbor of $v_1$ (even in $H$) this means that $\psi$ can be extended to $v_1$.

Now, consider $d_H(v_1) > 1$. First, suppose that there exists $c \in \mathcal{L}(v_1)$ such that $\{c-1, c+1\} \cap \mathcal{L}(v_1) = \emptyset$. Let $\mathcal{L}'$ be obtained by removing $c-1$ and $c+1$ from $\mathcal{L}(v_2)$, and $c$ from $\mathcal{L}(x)$ for every $x \in N(v_1) \setminus \{v_2\}$. Then $((H - v_1, P - v_1), \mathcal{L}')$ is a reduction of $((H, P), \mathcal{L})$ on $v_1$, and we want to show that it is reversible. So let $\psi$ be an $\mathcal{L}'$-CBC-7-coloring of $(H - v_1, P - v_1)$, and let $F = \psi(N(v_1)) \cup \langle \psi(v_2) \rangle$, the set of colors that are forbidden for $v_1$. If $\psi(v_2) \neq c$, we can color $v_1$ with $c$. Otherwise, we get $|\mathcal{L}(v_1) \cap \langle \psi(v_2) \rangle| = 1$, which implies that

$$|\mathcal{L}(v_1) \cap F| \leq |\mathcal{L}(v_1) \cap \psi(N(v_1) \setminus \{v_2\})| + |\mathcal{L}(v_1) \cap \langle \psi(v_2) \rangle| \leq d(v_1) - 1 + 1. \quad (6)$$

Since $\ell(v_1) = d_H(v_1) + 1$, there is a color in $\mathcal{L}(v_1) \setminus F$ with which we can color $v_1$.

Finally, suppose that

$$\{c-1, c+1\} \cap \mathcal{L}(v_1) \neq \emptyset, \text{ for every } c \in \mathcal{L}(v_1). \quad (\star)$$

Because $2 \leq d_H(v_1) \leq 4$ and $\ell(v_1) = d(v_1) + 1$, we have that $3 \leq \ell(v_1) \leq 5$. By $(\star)$ and the fact that we have 7 colors in total, there is a color $c$ such that $c \notin \mathcal{L}(v_1)$ and $\{c+1, c+2\} \subseteq \mathcal{L}(v_1)$. Without loss of generality, assume $c = 7$, so that $\{1, 2\} \subset \mathcal{L}(v_1)$ and that $7 \notin \mathcal{L}(v_1)$. We claim that we can also suppose (without loss of generality) that $6 \notin \mathcal{L}(v_1)$. Assume otherwise; by $(\star)$ we get that $\{1, 2, 5, 6\} \subseteq \mathcal{L}(v_1)$. If $\{3, 4\} \cap \mathcal{L}(v_1) = \emptyset$, then we rotate the colors so that 1 coincides with 5 and the desired property holds. Otherwise, we get a contradiction to Condition 3 of Lemma 12, since $|\langle c \rangle \cup \langle 7 \rangle| = 6$ where $c \in \{3, 4\} \setminus \mathcal{L}(v_1)$. Now, let $\mathcal{L}'$ be obtained by removing 1 from $\mathcal{L}(v_i)$ for every $v_i \in N(v_1) \setminus \{v_2\}$, and 1 and 2 from $\mathcal{L}(v_2)$, and let $\psi$ be an $\mathcal{L}'$-CBC-7-coloring of $(H - v_1, P - v_1)$. If $\psi(v_2) \neq 7$, we can color $v_1$ with 1. Otherwise, since $\{6, 7\} \cap \mathcal{L}(v_1) = \emptyset$, we get $|\langle \psi(v_2) \rangle \cap \mathcal{L}(v_1)| = 1$ and, again by inequality (6), we get that there must exist a color in $\mathcal{L}(v_1)$ with which we can color $v_1$. □

The sketch of the proof of Theorem 6 goes as follows. For the rest of this section we consider a planar graph $G$ with no cycles of length 4, a spanning linear forest $F$ of $G$, and assume that $(G, F)$ is a minimal counterexample to Theorem 6. We will further assume that $F$ has a component that contains a subpath $P$ with a particular set of properties (described latter). We know that, by minimality of $(G, F)$, there exists a CBC-7-coloring $\psi$ of $(G - P, F - P)$. In order to get a contradiction, it is enough to prove that there exists a $A_\psi$-CBC-coloring of $(G[V(P)], P)$, as by the definition of $A_\psi$ such a coloring extends $\psi$ to a CBC-7-coloring of $(G, F)$. For that aim, using the properties of $P$, we prove that we may iteratively apply Lemma 12 starting with the pair $((G[V(P)], P), A_\psi)$ until the reduced graph and path become a single vertex, and we show that the resulting list of colors available for such vertex shall have size at least one. This contradiction ensures that there can be no path $P$ with such particular set of properties in a minimal counterexample. Finally, we use this structural information about subpaths of $F$ to define our discharging rules.

The next lemma gives some trivial structural information about a minimal couterexample to Theorem 6.

**Lemma 13.** *Let $(G, F)$ be a minimal counterexample to Theorem 6. Then, we have $\delta(G) \geq 3$, and if $v \in V(G)$ is such that $d_G(v) \leq 4$, then $d_F(v) = 2$.*

*Proof.* This lemma follows easily from Lemma 7 (with $k = 7$) together with the fact that $F$ is a linear forest (and so every vertex has degree at most 2 in $F$). □

Before we present the types of paths that cannot occur in $F$ (where $(G, F)$ is defined as above), we need a further definition. Let $C$ be a component of $F$. If $P$ is a maximal subpath of $C$ containing only vertices of degree (in $G$) at most 5, we say that $P$ is a *heavy subpath of $C$*.

The next lemma, combined with Lemma 13, says that a heavy subpath can have only vertices of degree 5, except for either at most one vertex of degree 3 or at most two vertices of degree 4, and those exceptions cannot be a leaf of $F$.

**Lemma 14.** *Let $(G, F)$ be a minimal counterexample to Theorem 6, and $P$ be a heavy subpath of a component of $F$. All the following hold:*

*(a) If $P$ contains a leaf of $F$, then $d_G(u) = 5$, for all $u \in V(P)$;*

*(b) If $P$ has one vertex $v$ of degree 3, then $d_G(u) = 5$, for all $u \in V(P) \setminus \{v\}$;*

*(c) $P$ has at most two vertices of degree 4.*

*Proof.* Below, we consider a subpath $Q$ of $P$, and denote by $H$ the subgraph $G[V(Q)]$. We prove that whenever $P$ does not satisfy one of the assertions, then, letting $\psi$ be any CBC-7-coloring of $(G - H, F - H)$ (that exists by the minimality of $(G, F)$), we get that $(H, Q)$ is $A_\psi$-CBC-colorable, contradicting the fact that $(G, F)$ is a minimal counterexample to Theorem 6. We recall that, by Lemma 13, we have $\delta(G) \geq 3$ and $d_G(u) \geq 5$ whenever $d_F(u) \leq 1$.

First, suppose that either $(a)$ or $(b)$ does not hold, and let $Q = (v_1, v_2, \ldots, v_q)$ be a shortest subpath of $P$ such that $q \geq 2$, $d(v_1) \leq 4$, and either $d(v_q) = 3$ or $v_q$ is a leaf in $P$. Note that by definition of $Q$, the degree (in $G$) of any other vertex of $Q$ is equal to 5. Let $\psi$ be any CBC-7-coloring of $(G - H, F - H)$. For $i \in \{1, \ldots, q\}$, let $Q_i = (v_i, \ldots, v_q)$ and $H_i = H[\{v_i, \ldots, v_q\}]$.

We also let $A_1 = A_\psi$ and for $1 \leq i \leq q - 1$ we shall use Lemma 12 to obtain $A_{i+1}$ such that we have a reversible reduction of $((H_i, Q_i), A_i)$ on $v_i$ to $((H_{i+1}, Q_{i+1}), A_{i+1})$. For shortness, we denote $((H_i, Q_i), A_i)$ by $R_i$. In order to do that, we need to guarantee that the conditions to apply Lemma 12 are satisfied. For that aim, we show (inductively on $i$) that the following inequalities hold.

$$\ell_i(v_j) \geq d_{H_i}(v_j) + 2, \text{for every } j \text{ such that } i < j < q. \tag{7}$$

$$\ell_i(v_i) \geq d_{H_i}(v_i) + 1, \text{if } i < q. \tag{8}$$

$$\ell_i(v_q) \geq \begin{cases} d_{H_i}(v_q) + 2, & \text{if } i < q, \\ 1, & \text{otherwise.} \end{cases} \tag{9}$$

In particular, this shows that $A_q \neq \emptyset$. Observe that this leads to a contradiction since $Q_q = (v_q)$ and a coloring of $v_q$ with any color $c \in A_q$ can be extended to an $A_\psi$-CBC-coloring of $(H, Q)$ by the definition of reversible reduction. Denote by $\ell_i(v_j)$ the value $|A_i(v_j)|$, for each $i \leq j \leq q$. Consider the inequalities:

**Claim 15.** *If inequalities* (7), (8), *and* (9) *hold some* $i$*, with* $1 \leq i < q$*, and whenever* $i \neq 1$ *we additionally have that* $R_i$ *is a reduction of* $R_{i-1}$*, then we can apply Lemma 12 to* $R_i$ *and obtain* $R_{i+1}$*.*

*Proof of the claim.* Notice that the conditions of Lemma 12 concern $d_{H_i}(v_i)$ (not $d_G(v_i)$). Condition 1 of Lemma 12 applied to $R_i$ requires that $d_{H_i}(v_i) \leq 4$. This is clearly true, as $d(v_1) \leq 4$ and for every $2 \leq i < q$ we have $d(v_i) \leq 5$ and $v_i$ has a neighbor outside $H_i$ (namely $v_{i-1}$). Also, inequality (8) is equivalent to Condition 2 of Lemma 12. To check Condition 3 of Lemma 12 suppose that $d_{H_i}(v_i) = 4$. Recall that $d_G(v_1) \leq 4$ and $v_1$ is not a leaf of $Q$. Therefore, $i \neq 1$ and $d_H(v_i) = 5$. But since $d_{H_i}(v_i) = 4$, this means that $N_G(v_i) = N_{H_i}(v_i) \cup \{v_{i-1}\}$, which implies that $A_{i-1}(v_i) = [7]$ (as no neighbor of $v_i$ was previously colored, we still have all colors available for $v_i$). Then, Condition 3 of Lemma 12 follows by the fact that $R_i$ is a reduction of $R_{i-1}$. □

Continuing the proof of Lemma 14, we first argue that inequalities above hold for $i = 1$. Recall that $H_1 = H$, $P_1 = Q$, and $A_1 = A_\psi$. First, consider any $j \in \{2, \ldots, q-1\}$. Since $F$ is a linear tree, we have that $N_F(v_j) \subseteq Q$, which means every vertex not in $H$ forbids at most one color for $v_j$, therefore, $\ell_1(v_j) \geq 7 - d_{G-H}(v_j) = 7 - (d_G(v_j) - d_H(v_j))$. By the choice of $v_1$ and $v_q$, we know that $d_G(v_j) = 5$, which in turn implies inequality (7). Now, by Lemma 13, we know that $d_P(v_1) = 2$; so let $v \in N_P(v_1) \setminus \{v_2\}$. Note that $v$ forbids three colors for $v_1$, while each other colored neighbor of $v_1$ forbids just one color. This gives us that $\ell_1(v_1) \geq 7 - (d_{G-H}(v_1) + 2d_{P-Q}(v_1)) = 5 - (d_G(v_1) - d_H(v_1)) \geq d_H(v_1) + 1$. Analogously, for $v_q$ we get: if $d(v_q) = 3$, then $d_F(v_q) = 2$ and $\ell_1(v_q) \geq d_H(v_q) + 2$; and if $v_q$ is a leaf in $P$, then by Lemma 13 we get $d_G(v_q) = 5$, and as before $\ell_1(v_q) = 7 - d_{G-H}(v_q) \geq d_H(v_q) + 2$.

Now, suppose inequalities (7), (8), and (9) work for some $1 \leq i \leq q-1$. By Claim 15, there is a reversible reduction $R_{i+1}$ obtained from $R_i$. We want to prove that inequalities (7), (8), and (9) also hold for $R_{i+1}$. First, note that if $v_j \in N(v_i) \setminus \{v_{i+1}\}$, then $d_{H_{i+1}}(v_j) = d_{H_i}(v_j) - 1$ and, by definition of a reduction, $\ell_{i+1}(v_j) \geq \ell_i(v_j) - 1$. Hence, inequality (7) holds. Furthermore, in the case where $i \leq q-2$, inequality (9) also holds. Similarly, $d_{H_{i+1}}(v_{i+1})$ decreases by 1, while $\ell_{i+1}(v_{i+1})$ decreases by at most 2; hence, if $i \leq q-2$, we have that $\ell_i(v_{i+1}) \geq d_{H_i}(v_{i+1}) + 2$, which means that inequality (8) also holds for $R_{i+1}$. Finally, suppose that $i = q-1$. Then $\ell_{q-1}(v_q) \geq d_{H_{q-1}}(v_q) + 2 = 3$, and by the definition of reduction we get that $\ell_q(v_q) \geq 1$, i.e., inequality (9) holds also when $i = q-1$, and we are done proving $(a)$ and $(b)$.

Finally, in order to prove $(c)$, suppose for a contradiction that $d(x) = 4$, for three vertices $x \in V(P)$, and let $u, v, w \in V(P)$ be the closest three vertices of degree 4 in $P$, where $v$ is between $u$ and $w$. This time, we let $Q = (v_1, v_2, \ldots, v_q)$ be the subpath of $P$ from $u$ to $w$, so $v_1 = u$, $v_1 = w$, and we let $v_p = v$. As before, denote $G[V(Q)]$ by $H$, and let $\psi$ be a CBC-7-coloring of $(G - H, F - H)$. Note that, by cases $(a)$ and $(b)$, any vertex of $Q$ different from $v_1$, $v_p$ and $v_q$, must have degree 5 in $G$. Therefore:

- For each $z \in V(Q) \setminus \{v_1, v_p, v_q\}$, we get

$$a_\psi(z) \geq 7 - d_{G-H}(z) = 7 - (d_G(z) - d_H(z)) = 2 + d_H(z);$$

- For $z \in \{v_1, v_q\}$, we get $a_\psi(z) \geq 4 - (d_{G-H}(z) - 1) = 1 + d_H(z)$; and
- $a_\psi(v_p) = 7 - d_{G-H}(v_p) = 3 + d_H(v_p)$.

By analogous arguments to that of the first two cases, one can verify that a series of reversible reductions can be made on $Q$. First, for $1 \leq i \leq p$, we let $Q_i = (v_i, \ldots, v_p, \ldots, v_q)$, and we do $p-1$ until we obtain a triple $((H_p, Q_p), A_p)$. Later, we continue to shrink $Q_p$ but doing reductions from the other side, that is, we let $Q_p^j = (v_p, \ldots, v_{q-j})$, for $0 \leq j \leq q-p$, until we arrive at $Q_p^p = (v_p)$ together with some non-empty list $A_p^p$. As before, this is a contradiction. So, statement $(c)$ must hold. □

*5.2. Discharging Method*

Now, we are able to prove Theorem 6. We make an abuse of language and use the same nomenclature as in the proof of Theorem 5, although the terms "island" and "bad island" have have only a analogous meaning.

Consider a plane graph $G$ and its dual $G^*$, and let $F_3$ be the set of faces of degree 3 in $G$. In this section, we are assuming that $G$ contains no cycle of length 4, so all faces in $G^* - F_3$ have degree at least 5. Then, this time we denote the graph $G^* - F_3$ by $G_5^*$. We say that a component of $G_5^*$ is an *island of* $G$. Also, if $H$ is an acyclic component of $G_5^*$ such that $d_{G^*}(f) = 5$ for every $f \in V(H)$, then we say that $H$ is a *bad island of* $G$. We denote the set of bad islands of $G$ by $\Gamma$ and we let $\gamma$ denote $|\Gamma|$. Also, for $v \in V(G)$, we denote by $\Gamma(v)$ the set of bad islands containing $v$, and by $\gamma(v)$ the value $|\Gamma(v)|$. If $X \subseteq V(G)$, then $\Gamma(X) = \bigcup_{x\in X} \Gamma(x)$, and $\gamma(X) = |\Gamma(X)|$. In the remainder of the text, although we refer to $G$ as being planar, we are fixing a particular (planar) embedding of $G$ and its islands.

**Lemma 16.** *Let $G$ be a plane graph without cycles of length 4 as subgraph. Then*

$$\sum_{v\in V(G)} (d(v) - 4) \leq \frac{2\gamma}{3} - 8.$$

*Proof.* Note that the inequality in this lemma is equivalent to

$$m \leq 2n - 4 + \frac{\gamma}{3},$$

where $n$ denotes $|V(G)|$ and $m$ denotes $|E(G)|$.

Let $f_3, f_5$ denote the number of faces of degree 3 and 5, respectively. Also, denote by $\mathcal{F}$ the set of faces of $G$ and by $|f|$ the degree of a face $f \in \mathcal{F}$. We claim that:

$$3f_3 + f_5 \leq m + \gamma \tag{10}$$

This implies that $t = \sum_{f\in\mathcal{F}}(|f| - 6) \geq -3f_3 - f_5 \geq -m - \gamma$. On the other hand $t = \sum_{f\in\mathcal{F}}(|f|) - 6|\mathcal{F}| = 2m - 6|\mathcal{F}|$. Combining these and applying Euler's Formula we get:

$$2m - 6(2 - n + m) \geq -m - \gamma.$$

Therefore,

$$m \leq 2n - 4 + \frac{\gamma}{3}.$$

It remains to prove inequality (10). This is entirely analogous to the proof of Lemma 10, replacing $f_4$ by $f_5$ due to the fact that now we are working on $G_5^*$ instead of $G_4^*$ and to our definition of bad islands. We repeat it here for the sake of completeness. Partition $E(G)$ into $E_3 \cup \overline{E}_3$, where

$$E_3 = \{e \in E(G) : e \text{ is contained in some 3-face}\}.$$

Because $G$ has no cycle of length 4, we trivially get that $|E_3| = 3f_3$. To finish the proof, we show that $|\overline{E}_3| \geq f_5 - \gamma$. For this, note that if $e \in \overline{E}_3$ if and only if its dual edge $e^*$ belongs to $E(G_5^*)$. Therefore, $|\overline{E}_3| = |E(G_5^*)|$. Finally, letting $\mathrm{i}(G_5^*)$ be the number of acyclic components of $G_5^*$, we get:

$$|\overline{E}_3| \geq |V(G_5^*)| - \mathrm{i}(G_5^*) \geq f_5 - \gamma.$$

□

Supposing that $(G, F)$ is a minimal counterexample to Theorem 6, we apply the Discharging Method to prove that

$$\sum_{v \in V(G)} (d(v) - 4) - \frac{2\gamma}{3} \geq 0,$$

contradicting Lemma 16. We start by giving charge $d(v) - 4$ to every $v \in V(G)$, and charge $-2/3$ to every bad island. We will have five discharging rules to ensure that every vertex and every bad island ends up with a non-negative charge: every bad island will receive charge by some of the rules, and Property $P(i)$ below shall hold for every $i \in [5]$. Therefore, in the end, we conclude that no minimal counterexample exists and we are done with the proof of Theorem 6.

Given $x \in V(G) \cup \Gamma$, the initial charge of $x$ is denoted by $\mu_0(x)$, and the charge of $x$ after Rule $i$ is applied is denoted by $\mu_i(x)$, for each $i \in [5]$.

**Property** $P(i)$**.** *After Rule $i$ is applied, we have that $\mu_i(v) \geq 0$ and $\mu_i(b) \geq 0$ for every vertex $v$ every bad island $b$ whose charge has changed while applying Rule $i$.*

The rules are applied in the order they are presented. For each $i \in [5]$, after we state Rule $i$, we give a proof that Property $P(i)$ is satisfied.

**Rule 1.** *For every $v \in V(G)$ with $d(v) \geq 6$, send $2/3$ from $v$ to each $b \in \Gamma(v)$.*

*Proof of Property $P(1)$.* Consider $v \in V(G)$ with $d(v) \geq 6$. Because every island containing $v$ receives $2/3$, we just need to prove that $\mu_1(v) \geq 0$. Because $G$ has no cycles of length 4, observe that $\gamma(v) \leq \frac{d(v)}{2}$. This gives us that:

$$\mu_1(v) \geq d(v) - 4 - \frac{2}{3}\gamma(v) \geq d(v) - 4 - \frac{2}{3} \cdot \frac{d(v)}{2} \geq \frac{2}{3}d(v) - 4 \geq 0. \qquad (11)$$

□

The following proposition will be useful in the remainder of the text.

**Proposition 17.** *If $G$ is a graph without cycles of length 4, and $uv \in E(G)$, then there exists a face of degree greater than 3 containing $uv$.*

*Proof.* Observe that it holds because at least one of the two faces containing $uv$ cannot be a face of degree 3, as otherwise we would have a cycle of length 4. □

**Rule 2.** *For each heavy subpath $P = (v_1, \dots, v_q)$ (contained in a component of $F$) that has no vertex with degree smaller than 5, do:*

*R2.1 If $P$ is a component of $F$, send charge $2/3$ from $\mu_1(v_1) + \mu_1(v_2)$ to every $b \in \Gamma(\{v_1, v_2\})$. After this, if $q \geq 3$, then for each $i \in \{3, \dots, q\}$, send charge $2/3$ from $v_i$ to $b \in \Gamma(v_i) \setminus \Gamma(v_{i-1})$,*

*R2.2 Otherwise, let $v_0 \in N_F(v_1) \setminus \{v_2\}$. For every $i \in \{1, \dots, q\}$, send charge $2/3$ from $v_i$ to $b \in \Gamma(v_i) \setminus \Gamma(v_{i-1})$.*

*Proof of Property $P(2)$.* First, note that $\mu_1(v_i) = 1$ for every $i \in \{1, \dots, q\}$. Suppose that $P$ is a component of $F$. Note that Lemma 7 implies that $q \geq 2$. By Proposition 17, we get that $\gamma(\{v_1, v_2\}) \leq 3$, and that, when $q \geq 3$, then for every $i \in \{3, \dots, q\}$ we get $|\Gamma(v_i) \setminus \Gamma(v_{i-1})| \leq 1$. So, Property $P(2)$ follows.

Now, suppose that $P$ is not a component of $G$, in which case we can suppose, without loss of generality, $N_F(v_1) \setminus \{v_2\}$ is non-empty and let $v_0$ be one of its elements. By the definition of heavy path, we know that $d(v_0) \geq 6$, which, by Rule 1, implies that the island in $\Gamma(v_0) \cap \Gamma(v_1)$ has non-negative charge. Now, applying Proposition 17, for each $v_i \in V(P)$ we get that $|\Gamma(v_i) \setminus \Gamma(v_{i-1})| \leq 1$. Hence, Property $P(2)$ also follows in this case. □

**Rule 3.** *For each heavy subpath $P = (v_1, \dots, v_q)$ (contained in a component of $F$) that contains exactly one vertex with degree smaller than 5, say $v_p$, let $v_0 \in N_F(v_1) \setminus P$ and $v_{q+1} \in N_F(v_q) \setminus P$ and do one of the following:*

*R3.1 If $q \geq 2$, we can suppose that $p < q$, and:*

- *(i) Send charge $2/3$ from $v_i$ to $b \in \Gamma(v_i) \setminus \Gamma(v_{i-1})$, for each $i \in \{1, \dots, p-1\}$;*
- *(ii) Send charge $2/3$ from $v_i$ to $b \in \Gamma(v_i) \setminus \Gamma(v_{i+1})$, for each $i \in \{p+2, \dots, q\}$;*
- *(iii) If $d(v_p) = 3$, then $v_{p+1}$ sends charge 1 to $v_p$. Otherwise, $v_{p+1}$ sends charge $2/3$ to $b \in \Gamma(v_p) \cap \Gamma(v_{p+1})$.*

*R3.2 If $q = 1$ and $d(v_1) = 3$, let $b \in \Gamma(v_1)$. Send charge 1 from $\mu_2(v_0) + \mu_2(v_2) + \mu_2(b)$ to $v_1$.*

*Proof of Property $P(3)$.* By Lemma 14, we know that $v_0$ and $v_{q+1}$ exist, and, by Rule 1, we know that the islands in $\Gamma(v_0) \cap \Gamma(v_1)$ and $\Gamma(v_q) \cap \Gamma(v_{q+1})$ have non-negative charge. First, suppose that $q \geq 2$. By arguments similar to the ones in the previous demonstrations, one can see that the vertices in $\{v_1, \dots, v_{p-1}, v_{p+2}, \dots, v_q\}$, as well as the islands containing them, have non-negative charge. Also, note that, by Proposition 17, either $d(v_p) = 3$ and the only island containing $v_p$ also contains $v_{p-1}$ and $v_{p+1}$, or $d(v_p) = 4$ and the island in $\Gamma(v_p) \cap \Gamma(v_{p+1})$ is the only one that might not be satisfied yet. In either case, one can verify that the rule satisfies $v_p$ or the refered island, depending on the case.

Now, suppose that $q = p = 1$. If $d(v_1) = 4$, then $\Gamma(v_1) \subseteq \Gamma(v_0) \cup \Gamma(v_2)$ and nothing needs to be done; so suppose otherwise. First note that, because $d(v_1) = 3$, the island $b \in \Gamma(v_1)$ also contains $v_0$ and $v_2$. This means that $b$ has received charge from both $v_0$ and $v_2$ when Rule 1 is applied; hence $\mu_2(b) = 2/3$. We end the proof by showing that $\mu_2(v_2) = \mu_1(v_2) \geq 2/3$. Note that, since $d(v_1) = 3$ and because $G$ has no cycle of length 4, we can suppose that $v_1$ has no common neighbor with $v_2$. Therefore, if $d(v_2) = 6$, then $\gamma(v_2) = 2$, and applying the first part of inequality (11), we get that $\mu_2(v_2) = 6-4-4/3 = 2/3$. On the other hand, if $d(v_2) \geq 7$, we get $\mu_2(v) \geq 2/3$ by inequality (11). $\square$

In the next discharging rule, given $X \subseteq V(G)$, we denote $\sum_{v\in X} \mu_3(x)$ by $\mu_3(X)$.

**Rule 4.** *For each heavy subpath $P = (v_1, \ldots, v_q)$ (of a component of $F$) containing exactly two vertices with degree smaller than 5, namely $v_p$ and $v_q$, $p < q$, take $v_0 \in N_F(v_1) \setminus P$ and $v_{\ell+1} \in N_F(v_\ell) \setminus P$. Define*

$$\beta = \Gamma(V(P)) \setminus \Gamma(\{v_0, v_{\ell+1}\}),\ \textit{and}$$

$$\mu = \mu_3(V(P)) + \frac{2}{3}|\Gamma(v_0) \cap \Gamma(v_{\ell+1})|.$$

*If $\mu \geq \frac{2}{3}|\beta|$, then send $2/3$ from $V(P)$ and $\Gamma(v_0) \cap \Gamma(v_{\ell+1})$ to each $b \in \beta$.*

*Proof of Property $P(4)$.* The condition to apply Rule 4 already guarantees that Property $P(4)$ is satisfied. $\square$

We still need a final rule for the paths on which the condition $\mu \geq \frac{2}{3}|\beta|$, required in the previous rule, does not hold. Before we present the rule, we give sufficient conditions for Rule 4 to be applied. This is important because Rule 5 will be applied only after we cannot apply Rule 4 anymore.

**Lemma 18.** *If $P$ is a heavy subpath containing exactly two vertices with degree smaller than 5, and either $|V(P)| \geq 4$, or $\gamma(V(P)) \leq |V(P)|$, then $\mu \geq \frac{2}{3}|\beta|$.*

*Proof.* Consider $P, v_p, v_q, v_0, v_{\ell+1}, \beta, \mu$ be all defined as in Rule 4 (recall that $v_0, v_{\ell+1}$ exist by Lemma 14). First note that

$$|\beta| = \gamma(V(P)) - |\Gamma(V(P)) \cap \Gamma(\{v_0, v_{\ell+1}\}|.$$

Also, by Proposition 17, we have

$$\gamma(V(P)) \leq 2\ell - (\ell - 1) = \ell + 1.$$

Finally, by Lemma 14, we get that $d(v_p) = d(v_q) = 4$, and $d(v_i) = 5$ for every $v_i \in V(P) \setminus \{v_p, v_q\}$. Hence

$$\mu_3(V(P)) = \ell - 2.$$

Now, denote by $t$ the value $|\Gamma(V(P)) \cap \Gamma(\{v_0, v_{\ell+1}\})|$. By Proposition 17, we know that $t \geq 1$. We analyse the following cases:

- If $t = 1$, then the islands in $\Gamma(v_0) \cap \Gamma(v_1)$ and $\Gamma(v_\ell) \cap \Gamma(v_{\ell+1})$ must be the same, i.e., $\Gamma(v_0) \cap \Gamma(v_{\ell+1}) \neq \emptyset$, and $|\beta| = \gamma(V(P)) - 1$. Therefore,

  $$\mu \geq \mu_3(V(P)) + \frac{2}{3} = \ell - 2 + \frac{2}{3} = \ell - \frac{4}{3}.$$

  If $\ell \geq 4$, then $|\beta| \leq \ell$ and $\mu \geq \ell - \frac{4}{3} \geq \frac{2}{3}\ell \geq \frac{2}{3}|\beta|$. And if $\gamma(V(P)) \leq \ell$, then $|\beta| \leq \ell - 1$, and, since $\ell \geq 2$, we get $\mu = \ell - \frac{4}{3} \geq \frac{2}{3}(\ell - 1) \geq \frac{2}{3}|\beta|$.

- Now, if $t \geq 2$ and $\ell \geq 4$, then $|\beta| \leq \ell - 1$, and $\mu \geq \ell - 2 \geq \frac{2}{3}(\ell - 1) \geq \frac{2}{3}|\beta|$. Finally, if $t \geq 2$ and $\gamma(V(P)) \leq \ell$, then $|\beta| \leq \ell - 2$ and clearly $\mu \geq \ell - 2 \geq |\beta| \geq \frac{2}{3}|\beta|$.

□

Now, consider $P$ as in Rule 4 and suppose that the rule is not applied, which means that there might still exist some bad island intersecting $V(P)$ with negative charge. If such an island exists, we call such a path *defective*. Before we present the last discharging rule, we need the lemmas below. We mention that by Lemma 18, if $P$ is defective then $\ell \leq 3$ and $\gamma(V(P)) \geq \ell + 1$, where $\ell = |V(P)|$.

**Lemma 19.** *Let $P$ be a defective path of order $\ell$ with extremities $v_1$ and $v_\ell$, and denote by $v_2$ the neighbor of $v_1$ in $P$ (hence, it might happen that $\ell = 2$). Also, let $v_0 \in N_F(v_1) \setminus \{v_2\}$, and $v_{\ell+1} \in N_F(v_\ell) \setminus \{v_{\ell-1}\}$. Then, for each $i \in \{1, 2, \ell\}$, we have that $v_i$ is contained in exactly two bad islands (which means that $v_i$ is contained in two 3-faces that separate these bad islands), and $v_{i-1}v_{i+1} \notin E(G)$.*

*Proof.* First, suppose that $i \in \{1, 2, \ell\}$ is such that $v_i$ is contained in at most one triangle, which means that $\gamma(v_i) \leq 1$. Note that if $\ell = 3$, then $|\Gamma(v_1) \cap \Gamma(v_2) \cap \Gamma(v_3)| - |\Gamma(v_1) \cap \Gamma(v_3)| \leq 0$. This justifies the second line in the equation below.

$$\begin{aligned}
\gamma(V(P)) = \ & |\textstyle\bigcup_{v_j \in V(P)} \Gamma(v_j)| \\
\leq \ & \textstyle\sum_{j \in \{1,2,\ell\}} \gamma(v_j) - \sum_{j \in \{1,\ell-1\}} |\Gamma(v_j) \cap \Gamma(v_{j+1})| \\
\leq \ & \textstyle\sum_{j \in \{1,2,\ell\} \setminus \{i\}} \gamma(v_j) + \gamma(v_i) - (\ell - 1) \\
\leq \ & 2(\ell - 1) + 1 - \ell + 1 = \ell
\end{aligned}$$

This means that $P$ satisfies Lemma 18, a contradiction. Note also that this actually implies that each $v_i$ is contained in exactly two bad islands.

Now, suppose that $i \in \{1, 2, \ell\}$ is such that $v_{i-1}v_{i+1} \in E(G)$. Note that if $\ell = 3$ and $i = 2$, then $\gamma(V(P)) = \gamma(\{v_1, v_3\})$, and the island in $\Gamma(v_0) \cap \Gamma(v_1)$ also contains $v_3$. This implies that $\gamma(V(P)) = 3$, contradicting Lemma 18. So suppose, without loss of generality, that $i = 1$ and let $b$ be the island containing $v_0v_2$. Note that $\Gamma(v_1) \subseteq \Gamma(\{v_0, v_2\})$; therefore, $\beta = \Gamma(\{v_2, v_\ell\}) \setminus \Gamma(\{v_0, v_{\ell+1}\})$. First consider $\ell = 2$. If $b$ also contains $v_3$, then $|\beta| \leq |\Gamma(v_2) \setminus \{b\}| = 1$, and $\Gamma(v_0) \cap \Gamma(v_3) \neq \emptyset$, which implies $\mu \geq \frac{2}{3}|\beta|$. And if $b$ does not contain $v_3$, then $\Gamma(v_2) \subseteq \Gamma(\{v_0, v_3\})$, in which case $\beta = \emptyset$. Both cases are contradictions. Therefore, suppose that $\ell = 3$, and let $B$ denote $\Gamma(\{v_0, v_4\})$. Note that:

$$\begin{aligned}
|\beta| = \ & |\Gamma(\{v_2, v_3\}) \setminus B| \\
= \ & |(\Gamma(v_2) \setminus B) \cup (\Gamma(v_3) \setminus B)| \\
\leq \ & |(\Gamma(v_2) \setminus B)| + |(\Gamma(v_3) \setminus B)| \leq 2.
\end{aligned}$$

The last part holds since $b \in \Gamma(v_2) \cap B$, and $\Gamma(v_3) \cap \Gamma(v_4) \neq \emptyset$ (Proposition 17). If $|\beta| \leq 1$ we are done since $\mu \geq 1$. Therefore, suppose $|\beta| = 2$, in which case we must have $(\Gamma(v_2) \setminus B) \cap (\Gamma(v_3) \setminus B) = \emptyset$. So, let $b_i \in \Gamma(v_i) \setminus B$, for $i = 2$ and $i = 3$, and let $b^* \in \Gamma(v_3) \cap \Gamma(v_4)$. Because $\Gamma(v_2) \cap \Gamma(v_3) \neq \emptyset$ and $b_2 \neq b_3$, we get $b = b^*$, i.e., $b \in \Gamma(v_0) \cap \Gamma(v_4)$. Therefore, we get $\mu \geq 1 + \frac{2}{3} > \frac{4}{3} = \frac{2}{3}|\beta|$, a contradiction. □

The next lemma is the final step before we can present the last discharging rule. We denote by $\Theta$ the set of bad islands with negative charge, and by $D$ the set of vertices of degree 5 which are contained in some island in $\Theta$.

**Lemma 20.** *Let $b \in \Theta$, and $f$ be a face of degree 5 in $b$. Then $f$ contains at least one vertex of $D$ and, if it contains exactly one such vertex, namely $u$, then $b$ is the only island in $\Theta$ that contains $u$.*

*Proof.* Let $f = (v_1, \ldots, v_5)$ be such that $v_i$ is contained in some defective path, for each $i \in \{1, \ldots, 5\}$. Without loss of generality, suppose that $d(v_i) = 4$ for every $i \in \{1, \ldots, 4\}$. First, we want to prove that $(v_1, \ldots, v_5)$ is an induced cycle in $G$. So suppose that $v_1 v_3 \in E(G)$. Since $f$ is a 5-face in $G$, we must have that the edge $v_1 v_3$ is traced in the outer side of $f$. Because $\delta(G) \geq 3$, one can verify that this implies that $(v_1, v_2, v_3)$ is not a 3-face in $G$, which in turn implies that $v_1$ is contained in at most one bad island, contradicting Lemma 19. Observe that the same argument can be applied to conclude that $v_i v_j \notin E(G)$ for every $i \in \{1, \ldots, 4\}$ and every $j \in \{1, \ldots, 5\} \setminus \{i\}$. Now observe that, by Lemma 19, there must exist $u_1, \ldots, u_5$, where $u_5 \in N(v_1) \cap N(v_5)$, and $u_i \in N(v_i) \cap N(v_{i+1})$, for each $i \in \{1, \ldots, 4\}$. This means that every island in $\Theta$ is a face of degree 5. We claim that $d(v_5) = 5$. Supposing it holds, let $w \in N(v_5) \setminus \{v_1, v_4, u_4, u_5\}$; also let $f_1$ be the face containing $u_4 v_5$ different from $(v_4, v_5, u_5)$, and $f_2$ be the face containing $u_5 v_5$ different from $(u_5, v_5, v_1)$. Because $G$ has no cycles of length 4, we know that $f_1$ and $f_2$ have degree bigger than 3, and that they share the edge $v_5 w$. This means that $f_1$ and $f_2$ are within the same island $t$, which implies that $t \notin \Theta$, and the lemma follows, i.e., $b$ is the only island in $\Theta$ containing $u$. It remains to prove our claim.

Suppose by contradiction that $d(v_5) = 4$, and let $H$ denote the induced subgraph $G[\{v_1, \ldots, v_5, u_1, \ldots, u_5\}]$. Because $d_F(v_i) = 2$ and $N(v_i) \subseteq V(H)$ for every $i \in \{1, \ldots, 5\}$, we know that $H$ must contain every edge in $F$ incident to $\{v_1, \ldots, v_5\}$. For each $v_i$, let $E_i$ denote the set $\{uv_i \in E(F)\}$; we know that $|E_i| = 2$. Therefore, if $E_i \cap E_j = \emptyset$ for every $i, j \in \{1, \ldots, 5\}$, $i \neq j$, then $|E(H) \cap E(F)| = |\bigcup_{i=1}^{5} E_i| = \sum_{i=1}^{5} |E_i| = 10 = |V(H)|$, contradicting the fact that $F$ is acyclic. We can then suppose, without loss of generality, that $v_1 v_2 \in E(F)$. By Lemmas 14 and 19, we get that $\{u_5 v_1, u_2 v_2\} \subseteq E(F)$. Also, by Lemma 19, we get $|\{v_3 v_4, v_3 u_3\} \cap E(F)| \leq 1$ and $|\{v_4 v_5, u_4 v_5\} \cap E(F)| \leq 1$. This implies that $\{u_5 v_5, u_2 v_3\} \subseteq E(F)$. It is easy to verify that no matter the choice of edges in $E_4$, we get a cycle in $F$, a contradiction. □

**Rule 5.** *Let $K = (D, E)$ be such that $uv \in E$ if and only if $u$ and $v$ are within the same bad island $b \in \Theta$. For each component $K'$ of $K$, apply one of the following:*

*R5.1 If $|V(K')| \geq 2$, let $T$ be a spanning tree of $K'$ and let $uv \in E(T)$. Send charge $2/3$ from $\{u, v\}$ to each island in $\Gamma(\{u, v\})$, and for every $w \in$*

$V(T) \setminus \{u, v\}$, *send charge* $2/3$ *from* $w$ *to the island in* $\Gamma(w) \setminus \Gamma(w')$, *where* $w' \in N_T(w)$ *separates* $w$ *from* $uv$.

*R5.2 If* $V(K') = \{u\}$, *send* $2/3$ *from* $u$ *to the bad island in* $\Theta$ *containing* $u$.

*Proof of Property* $P(5)$. The fact that $P(5)$ holds follows from Lemma 20. □

Lemma 20 also guarantees that every bad island has received some charge after Rule 5. With this, we conclude the proof of Theorem 6.

## 6. Concluding Remarks and Further Research

In this article, we investigate Conjectures 2 and 3, and Questions 1 and 2. Our results have some kind of graduation from one to the other. Consider $G$ to be a planar graph, $M$ to be a matching of $G$, and $F$ to be a linear forest of $G$. In Theorem 4, we prove that if $G$ has no $C_4$ nor $C_5$, then $\mathrm{CBC}(G, M) \leq 5$. In Theorem 5, we forbid faces of size 3 of sharing an edge (graphs that do not contain $C_4$ satisfy that property), but we need one extra color, i.e., $\mathrm{CBC}(G, M) \leq 6$. Finally, in Theorem 6, we forbid only $C_4$ and allow for bigger backbones, namely linear forests, and once again we need an extra color, i.e., $\mathrm{CBC}(G, T) \leq 7$.

Each of our proofs uses the discharging method, and greatly profit from the fact that the investigated planar graphs do not have small cycles and that the backbones have simple structure. In particular, it seems to us that the method cannot be applied in general, since when nested triangles are allowed, they put a great burden in the discharging phase. This means that in order to generalize Theorem 5, another approach should be needed. Nevertheless, we believe that the discharging method can be applied to generalize Theorem 6 to general forest backbones.

Concerning Theorem 4, it is presented as a partial answer to Conjecture 3 and to Question 2. In fact, the conjecture had already been proved for linear forests in [13]. However, our theorem actually gives a better upper bound. Since Steingberg's Conjecture has been disprove, we cannot relax the condition on $H$ in Conjecture 3 to let $H$ be any spanning graph of $G$ (as this conjecture is equivalent to Steingberg's Conjecture when $H = G$). However, one may still ask if $H$ can be "larger" than a spanning tree in some sense.

Finally, recall that the original questions regarding matching and tree backbones are actually about the non-circular version of the problem (Conjecture 1 and Question 2). The proposed upper bounds are inspired by known examples where they are met. Namely, in [4] it is presented a planar graph $G$ and a tree $T$ of $G$ such that $\mathrm{BBC}(G, T) = 6$, and in [12] it is presented a planar graph $G$ and a matching $M$ of $G$ such that $\mathrm{BBC}(G, M) = 5$. However, in both examples it is possible to obtain circular backbone colorings that do not increase the number of used colors. Therefore, we ask whether the bounds for the circular case need to be larger.

**Question 3.** *Does there exist a pair* $(G, H)$ *such that* $G$ *is a planar graph,* $H$ *is a forest of* $G$, *and* $\mathrm{CBC}(G, H) = 7$? *Also, does there exist a pair* $(G, H)$ *such that* $G$ *is a planar graph,* $M$ *is a matching of* $G$, *and* $\mathrm{CBC}(G, M) = 6$?

One can also ask whether the bounds are tight in the particular cases investigated in this article.

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