THE NUMBER OF GALLAI k-COLORINGS OF COMPLETE GRAPHS

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Abstract. An edge coloring of the n-vertex complete graph, K_n, is a *Gallai coloring* if it does not contain any rainbow triangle, that is, a triangle whose edges are colored with three distinct colors. We prove that for n large and every k with $k \leq 2^{n/4300}$, the number of Gallai colorings of K_n that use at most k given colors is $\left(\binom{k}{2} + o_n(1)\right) 2^{\binom{n}{2}}$. Our result is asymptotically best possible and implies that, for those k, almost all Gallai k-colorings use only two colors. However, this is not true for $k \geq 2^{n/2}$.

§1. Introduction

An edge coloring of the complete graph on n vertices, K_n, is a *Gallai coloring* if it contains no *rainbow* K_3, that is, no copy of K_3 in which all edges have distinct colors. Here, we always use k-coloring to refer to an edge coloring that uses (not necessarily all) colors from a set of k colors; in contrast, for $\ell \leq k$, we say a graph G is ℓ-colored, if its edges are colored with *exactly* ℓ colors (out of k given colors).

The term *Gallai coloring* was used by Gyárfás and Simonyi in [15], though those colorings have also been studied under the name *Gallai partitions* by Körner, Simonyi and Tuza in [19]. The terminology is due to a close relation to a result in Gallai's influential original paper [12] – translated to English with added comments in [13]. The above mentioned papers are mostly concerned with structural and Ramsey-type results about Gallai colorings.

Following a recent trend of working on problems about counting certain colorings and analyzing the typical structure of them, for integers $k \geq 1$ and $n \geq 2$, we are interested in the problem of counting the number, $c(n, k)$, of Gallai k-colorings of K_n (that use a given set of k colors). In this counting, we consider the vertices of K_n as labeled.

The related problem of counting colorings of graphs such that every color class does not contain a particular graph F was studied originally by Erdős and Rothschild (see, e.g., [8]) and has motivated a number of results (for example [1, 16, 17, 20]) and it was generalized to colorings that avoid other coloring patterns in [4]. In turn, Gallai colorings have surprising relations to Information Theory [18] and a generalization of the (weak) Perfect Graph Theorem [5], and has

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also been generalized to non-complete graphs [14] and hypergraphs [7]. Recently, there has also been a trend in Ramsey-type problems that involve Gallai colorings (see [6, 10, 11, 21, 22]).

In this context, the problem of counting Gallai colorings is a very natural one and has been "almost" overlooked. A trivial lower bound for $c(n,k)$, when $k \geq 2$, is given by the colorings that use at most two colors: $c(n,k) \geq \binom{k}{2}(2^{\binom{n}{2}} - 2) + k$.

In [3], its was proved that $c(n,3) \leq 7(n+1)\, 2^{\binom{n}{2}}$, for every $n \geq 2$. Previous bounds for $c(n,3)$ were obtained by Falgas-Ravry, O'Connell, Strömberg and Uzzell [9], of order $2^{(1+o_n(1))\binom{n}{2}}$ using the container and entropy method, and around the same time Benevides, Hoppen and Sampaio [4] gave a bound of order $(n-1)!\, 2^{\binom{n}{2}}$.

The main purpose of this paper is to prove that the lower bound for $c(n,k)$ is asymptotically sharp for $k \leq 2^{n/4300}$. We point out that, in [15], it was proved that any Gallai coloring of K_n uses at most $n-1$ colors. In spite of this, the definition of $c(n,k)$ makes sense even for $k \geq n$, as in a Gallai k-coloring we do not have to use all k colors.

Theorem 1. *For n large enough and every k with $2 < k \leq 2^{n/4300}$, we have*

$$c(n,k) = \left(\binom{k}{2} + o_n(1)\right) 2^{\binom{n}{2}},$$

where $o_n(1)$ tends to zero exponentially fast as n tends to infinity.

This implies that almost all Gallai k-colorings of K_n use only two colors, for $k \leq 2^{n/4300}$. In other words, in this range, the typical structure of a Gallai k-coloring is simply that of a two-coloring. On the other hand for $k \geq 2^{n/2}$, the statement of Theorem 1 would not be true. To prove this, we consider another type of colorings that we call *star colorings* (also called *lexicographic colorings*). For any ordering of the vertices, say $v_1, \ldots, v_n$, and a fixed $(n-1)$-tuple of colors, say $(c_1, \ldots, c_{n-1})$, for every $i < j$, color the edge $v_i v_j$ with color c_i. For an easy lower bound on $c(n,k)$, to avoid a messy double counting, we count only the star colorings such that $c_1, \ldots, c_{n-1}$ are distinct. Thus, for $n \geq 4$, we have,

$$c(n,k) \geq \frac{n!}{2}\, k(k-1)\ldots(k-n+2) \geq k^{n-1}.$$

(above we have divided the $n!$ by 2, because swapping v_{n-1} with v_n yields the same colorings). For $k \geq 2^{n/2}$, we have $c(n,k) \geq 2^{n(n-1)/2} = 2^{\binom{n}{2}}$ (so it is already false that most Gallai colorings use only 2 colors). Moreover, for $k \gg 2^{n/2}$, the proportion of Gallai colorings that use at most 2 colors tends to zero.

In our proofs, the $o_n(1)$ term in Theorem 1 can be taken exponentially small in n, as in $o_n(1) \leq 2^{-n/150}$. In other to keep the calculation clean, we did not try to optimize either the $o_n(1)$ or the constant $1/4300$ in its statement. The same method could be fine tuned to yield a larger constant, but probably still far from (the best possible target) $1/2$.

Our proof is self contained with the exception of two (elementary) results from [15] (whose proofs are also short, we encourage the reader to look at them). Furthermore, our proofs are elementary, based on how to classify the colorings while counting the number of ways to extend them. It came recently to our attention that, independently, Balogh and Li [2] proved a similar upper bound, but for k constant and n large using the container and stability method.

In this paper, we also show (see Corollary 7) that

$$c(n,k) \le (k-1)^n \, 2^{\binom{n}{2}},$$

for every $n \ge 2$ and $k \ge 2$.

The following two results from [15] shall be useful to us.

Theorem 2 (see Theorem 2.1 or 2.2 of [15])**.** *Every Gallai coloring of a complete graph K_n contains a monochromatic spanning tree.*

Theorem 3 (Theorem 3.1 of [15])**.** *Every Gallai coloring of K_n has a color with maximum degree at least $2n/5$.*

In this paper all logarithms are on base 2. We omit the floor and ceiling functions as long as they do not affect the calculations. For any natural number n, we denote $\{1,\dots,n\}$ by $[n]$. Moreover, whenever we talk about three colors we refer to them as red, green and blue.

§2. Proof of the main result

Let $\Phi_{n\to k}$ be the set of all Gallai colorings of K_n that use colors in $[k]$. So, each element of $\Phi_{n\to k}$ is a function $\varphi : E(K_n) \to [k]$ and $c(n,k) = |\Phi_{n\to k}|$. For any Gallai coloring φ of $E(K_n)$, we denote by $w(\varphi,k)$ the number of ways to extend φ to a Gallai coloring of $E(K_{n+1})$ where the new edges receive colors from $[k]$ (regardless of how many colors actually appear in φ).

We start with the following trivial fact that calculates the number of extensions of a monochromatic coloring. This fact and Lemma 5 are generalizations of lemmas from [3].

Fact 4. *Let k and n be positive integers and $\varphi \in \Phi_{n\to k}$ be a monochromatic coloring of the edges of K_n. Then,*

$$w(\varphi,k) = (k-1)2^n - (k-2).$$

Proof. Without loss of generality, assume that all edges of K_n are colored blue. Let $\{u\} = V(K_{n+1}) \setminus V(K_n)$. Notice that in any extension of this coloring we can only use one color different from blue. So, for each choice of the other color (among the other $k-1$ available), we have 2^n extensions. As the extension in which all edges are blue is counted $k-1$ times, the total number of extensions is $(k-1)2^n - (k-1) + 1$, as claimed. □

Lemma 5. *Let k and n be positive integers and $\varphi \in \Phi_{n\to k}$ be any Gallai k-coloring of K_n. If $\varphi' \in \Phi_{n+1\to k}$ is an extension of φ to $E(K_{n+1})$, then*

$$w(\varphi',k) \le 2w(\varphi,k) + (k-2).$$

Proof. Let $V = V(K_n)$. Let φ' be an extension of φ to $E(K_{n+1})$ with at most k colors and $u \notin V$ be the new vertex (added to obtain K_{n+1}). To count the number of Gallai extensions of φ' to $E(K_{n+2})$, we will add a new vertex x and all edges from x to $V \cup \{u\}$. We first color the edges from x to V. If we let $t = w(\varphi, k)$, there are t colorings, say $\varphi_1, \ldots, \varphi_t$, of the edges from x to V. For each $i \in [t]$, we let m_i be the number of ways we can color the edge ux given that we have colored the edges from x to V as in φ_i. Clearly, $m_i \in \{0, 1, \ldots, k\}$ and $w(\varphi', k) = \sum_{i=1}^{t} m_i$.

Fix any $i \in [t]$. Recall that the edges from u to V are already colored (in φ'). If there is any vertex $v \in V$ such that $\varphi'(xv) \neq \varphi_i(uv)$, then the only colors that can be used for ux are $\varphi'(xv)$ and $\varphi_i(uv)$, then $m_i \leq 2$. Therefore, the only way to have $m_i \geq 3$ is when the coloring φ_i is such that $\varphi_i(xy) = \varphi'(uy)$ for every y in V, and for such coloring we have $m_i = k$. This implies that $\sum_{i=1}^{t} m_i \leq 2(t-1) + k = 2t + (k-2)$. □

We remark that when φ is a monochromatic coloring and φ' is its monochromatic extension, by Fact 4, we have $w(\varphi, k) = (k-1)2^n - (k-2)$ and $w(\varphi', k) = (k-1)2^{n+1} - (k-2)$. Therefore, $w(\varphi', k) = 2w(\varphi, k) + (k-2)$, which implies that Lemma 5 is best possible.

2.1. **Extensions of graphs that use exactly one color.** We start by proving an easy lemma that says that all Gallai k-colorings of K_n have at most as many extensions to a Gallai k-coloring of K_{n+1} as the monochromatic colorings. This also implies our general (but weak) upper bound on $c(n,k)$. As we shall see later, most of the Gallai k-colorings have significantly less number of extensions than the monochromatic ones, and this fact will imply a better bound for $c(n,k)$. However, the weak upper bound will also be useful in our proof.

Lemma 6. *For positive integers k and n, and every Gallai coloring $\varphi \in \Phi_{n \to k}$, we have*

$$w(\varphi, k) \leq (k-1)2^n - (k-2).$$

Proof. The proof is by induction on n. The result clearly holds for $n = 1$ and $n = 2$. Now, assume that $n \geq 2$, and $w(\varphi, k) \leq (k-1)2^n - (k-2)$ for every $\varphi \in \Phi_{n \to k}$. Let φ' be any Gallai k-coloring of K_{n+1}. Take any vertex $v \in V(K_{n+1})$ and let φ be the restriction of the coloring φ' to the edges of $K_{n+1} - v$. Lemma 5 implies that

$$w(\varphi', k) \leq 2w(\varphi, k) + (k-2) \leq 2((k-1)2^{n-1} - (k-2)) + (k-2) = (k-1)2^n - (k-2). \qquad \square$$

Remark. *It also follows from an analogous induction argument that the only colorings that achieve this maximum are the monochromatic ones, but we will not need this fact.*

Corollary 7. *For integers $n, k \geq 2$ we have $c(n,k) \leq (k-1)^n \, 2^{\binom{n}{2}}$.*

Proof. For each fixed $k \geq 2$, we use induction on $n \geq 2$. For the base case, when $n = 2$, we are simply saying that $k \leq (k-1)^2 \cdot 2^1$, which is true for $k \geq 2$. Now, assuming the result holds for some $n \geq 2$, by Lemma 6, we have

$$c(n+1,k) = \sum_{\varphi \in \Phi_{n \to k}} w(\varphi, k) \leq (k-1)2^n \cdot c(n,k) \leq (k-1)^{n+1} \, 2^{\binom{n+1}{2}}. \qquad \square$$

2.2. **Colorings that are rare or have few extensions.** Due to the results of the previous section, one may hope that the k-colorings that are "close to being monochromatic" are those that have the most extensions. Our proof of Theorem 1 does *not* require us to prove precisely this. But we do show that whenever we have a 2-*coloring* where both colors are used many times, then we have few extensions.

Lemma 8. *For every $s, m, k \geq 2$, every 2-coloring φ of K_m such that both colors are used more than $3sm$ times satisfies $w(\varphi, k) \leq 2^m + km2^{m-0.4s}$. Furthermore, at most $km2^{m-0.4s}$ such extensions use a color that is not used in φ.*

Proof. For $k = 2$ the result is trivial, since there are at most 2^m extensions. Let $k \geq 3$. Let φ be a coloring of $E(K_m)$ as in the statement with colors red and blue. As before, let $u \in V(K_{m+1}) \setminus V(K_m)$, so that we want to count the number of ways to color the edges from u to $V(K_m)$. First, note that there are 2^m ways to extend φ to a Gallai coloring of K_{m+1} using only red and blue. Secondly, note that using two new colors (that is, different from red and blue) will immediately create a rainbow triangle. Thus, it remains to show that there are at most $km2^{m-0.4s}$ extensions that use exactly one new color, say green.

We claim that there exists $\{(r_1, u_1, b_1), (r_2, u_2, b_2), \ldots, (r_{s+1}, u_{s+1}, b_{s+1})\}$, a set of disjoint triples of vertices of K_m such that $\varphi(u_i r_i) = \text{red}$ and $\varphi(u_i b_i) = \text{blue}$ for all $i \in [s+1]$. Indeed, assume that the maximum set of such disjoint triples has size s. This implies that all edges inside $V(K_m) - \bigcup_{i=1}^{s}\{r_i, u_i, b_i\}$ have the same color, say blue. So, the number of red edges is at most $3sm$, a contradiction.

Fix a vertex $v \in V(K_m)$. Suppose that we want to build an extension φ' of φ such that $\varphi'(uv) = \text{green}$. Let us count in how many ways we can complete the extension φ'. Note that, for any vertex $z \in V(K_m) \setminus \{v\}$, (as $\varphi(vz)$ is not green) $\varphi'(uz)$ must be either green or $\varphi(vz)$. So, assuming $\varphi'(uv) = \text{green}$, there are at most 2 choices for every other edge. However, we claim that for every $i \in [s+1]$ such that $v \notin \{u_i, r_i, b_i\}$, there are at most 6 ways to color the set of edges $\{uu_i, ur_i, ub_i\}$.

To see this, fix i such that $v \notin \{u_i, r_i, b_i\}$ and assume that $\varphi(vu_i) = \text{red}$ (the case $\varphi(vu_i) = \text{blue}$ is analogous). Recall that ub_i must receive either green or $\varphi(vb_i)$. In the case ub_i is green, then uu_i must also be green (looking at the triangles uvu_i and ub_iu_i) and we have (at most) 2 options for the color of ur_i. In the case ub_i has color $\varphi(vb_i)$, we trivially have at most 4 options for the colors of uu_i and ur_i. This gives a total of at most 6 ways to color the set $\{uu_i, ur_i, ub_i\}$.

Finally, we count how many extensions of φ use one new color (with some room to spare). We have m options to choose a vertex v, have $k-2$ options for the color of uv, have 6^s ways to color the edges from u to s of those $\{u_i, r_i, b_i\}$ such that $v \notin \{u_i, r_i, b_i\}$ and 2 ways to color each of the $m - 3s - 1$ remaining edges. This gives less than

$$km6^s 2^{m-3s-1} = km2^{m-(3-\log 6)s-1} \leq km2^{m-0.4s}$$

such extensions. ∎

Let $F(n,k)$ be the set of colorings of K_n with colors from $[k]$, such that one color forms a spanning subgraph of K_n and the others span pairwise vertex-disjoint (possibly empty) subgraphs of K_n. Clearly, for a coloring in $F(n,k)$, there is a partition of $V(K_n)$, say $V_1 \cup V_2 \cup \ldots \cup V_{k-1}$, and a color, say red, such that for all $i, j \in [k]$ with $i \neq j$ all edges from V_i to V_j are red, and V_i induces a clique that uses at most two colors (one of which is red). Let $F'(n,k) \subseteq F(n,k)$ be the set of such colorings with the extra assumption that there is no set of at least $0.9n$ vertices that induces a 2-colored clique. Let $f'(n,k) = |F'(n,k)|$.

Remark 9. *By Theorem 2, a Gallai k-coloring of K_n is in $F(n,k)$ if, and only if, it does not have a vertex that has three edges of different colors incident to it.*

Next, we give an upper bound on $f'(n,k)$ (that is, we show that this type of colorings are rare).

Lemma 10. *For every $n, k \geq 2$, we have*

$$f'(n,k) \leq 2^{\binom{n}{2} - 0.05n^2 + (n+1)\log k}.$$

Proof. There are k choices for the color that forms a spanning subgraph. Assume, without loss of generality that we have chosen color k for it. There are $(k-1)^n < k^n$ ways to partition the vertices of K_n into (labeled and possibly empty) $k-1$ classes. For $i \in [k-1]$, let x_i be the number of vertices in class i. We have that $\sum_{i \in [k-1]} x_i = n$ and there are $2^{\sum_{i\in[k-1]} \binom{x_i}{2}}$ ways[1] to color the edges of K_n so that those inside class i receive color i or color k, for every $i \in [k-1]$, and edges between classes receive color k. Note that

$$\sum_{i\in[k-1]} \binom{x_i}{2} = \binom{n}{2} - \sum_{1 \leq i < j \leq k-1} x_i x_j = \binom{n}{2} - \frac{1}{2} \sum_{i \in [k-1]} x_i(n - x_i).$$

By the definition of $F'(n,k)$ we have $x_i \leq 0.9n$ for every $i \in [k-1]$. Therefore,

$$\sum_{i\in[k-1]} x_i(n-x_i) \geq \sum_{i \in [k-1]} x_i(0.1n) = 0.1n^2.$$

Thus,

$$\sum_{i\in[k-1]} \binom{x_i}{2} \leq \binom{n}{2} - 0.05n^2.$$

In total, we get

$$f'(n,k) \leq k \cdot k^n 2^{\binom{n}{2} - 0.05n^2} = 2^{\binom{n}{2} - 0.05n^2 + (n+1)\log k}.$$ □

The next lemma treats another case in which we can guarantee that a Gallai k-coloring has few extensions: when it has no "large" set of vertices S that induces a k-coloring in $F(|S|, k)$. For k in the range of Theorem 1, the number of such extensions is significantly less than the one for monochromatic colorings.

[1]For the sake of this notation, we are considering $\binom{0}{2} = \binom{1}{2} = 0$.

Lemma 11. *Let $k, t, n \geq 2$ be integers such that $n \geq 75t$, and $\varphi \in \Phi_{n \to k}$ be a Gallai k-coloring that does not contain any set of $n/3$ vertices which induce a coloring in $F(n/3, k)$. Then*

$$w(\varphi, k) \leq tk^2 2^{n-t}.$$

Proof. Let φ be a Gallai k-coloring of K_n as in the statement of this lemma. We first prove that φ must contain a certain structure. By Theorem 3, we can inductively choose vertices $v_1, \dots, v_t$, such that for every $i \in [t]$, there is a color c_i such that v_i is adjacent to at least $2(n-i+1)/5 \geq 2n/5 - t$ vertices in $K_n \setminus \{v_1, \dots, v_{i-1}\}$ through edges of color c_i. Let $T = \{v_1, \dots, v_t\}$.

Next, for each $i \in [t]$, we will build a family $\mathcal{F}_i$ of t vertex-disjoint rainbow copies of $K_{1,3}$ (where different copies may use different triples of colors), each contained in $N_{c_i}(v_i) \setminus T$, where $N_{c_i}(v_i)$ stands for the set of vertices that are connected to v_i via edges of color c_i. Moreover, for $i \neq j$, we do not care whether the copies of $K_{1,3}$ in $\mathcal{F}_i$ are disjoint from those in $\mathcal{F}_j$.

Fix $i \in [t]$. We will build $\mathcal{F}_i$ greedily. Suppose we have added less than t copies of $K_{1,3}$ to $\mathcal{F}_i$ and we want to find an extra one. In total, the current copies take at most $4t$ vertices. Thus, there are at least $|N_{c_i}(v_i)| - 4t \geq 2n/5 - |T| - 4t = 2n/5 - 5t \geq n/3$ vertices in $N_{c_i}(v_i) \setminus T$ that do not belong to any $K_{1,3}$ in $\mathcal{F}_i$. Let A be the set of those vertices. If any of them is incident to 3 edges of distinct colors with all endpoints also in A, then we can add another $K_{1,3}$ to $\mathcal{F}_i$ and we are done. Otherwise, by Remark 9, applied to the coloring induced by A, those vertices in A induce a coloring in $F(|A|, k)$. But $|A| \geq n/3$, contradicting the hypothesis in Lemma 11.

Now, we prove the upper bound for $w(\varphi, k)$. Consider a new vertex u and let us count in how many ways we may color the edges from u to K_n.

First consider the case where for some $i \in [t]$, the edge uv_i receives a color different from c_i, say $c_{u,i}$. Then consider the edges in the rainbow $K_{1,3}$'s contained in $N_{c_i}(v_i)$ that have a color different from c_i and $c_{u,i}$. Each $K_{1,3}$ has at least one such edge, therefore, we can select a matching M of size t formed by those edges. Let us denote it by $M = \{a_1b_1, \dots, a_tb_t\}$. By considering the quadruple u, v_i, a_j, b_j, for $j \in [t]$, it is easy to see that the colors of ua_j and ub_j must be equal and be either c_i or $c_{u,i}$. Thus, there are at most 2^t ways to color the $2t$ edges ua_j and ub_j, where $j \in [t]$. It remains to color the edges from u to S where $S = V(K_n) \setminus (V(M) \cup \{v_i\})$. Applying Lemma 6 (to count extensions of $K_n[S]$), we conclude that there are less than $(k-1)2^{n-2t-1}$ ways to colors those edges. Summing over all $i \in [t]$ and the choice of the color of uv_i, we obtain at most

$$t \cdot (k-1) \cdot 2^t \cdot (k-1)2^{n-2t-1} = t(k-1)^2 \, 2^{n-t-1}$$

extensions in this case.

It remains to consider the case where for all $i \in [t]$, the edge uv_i receives color c_i. In this case, we let $S = V(K_n) \setminus \{v_1, \dots, v_t\}$ and by Lemma 6, we can color the edges from u to S in less than $(k-1)2^{n-t}$ ways.

In total, adding both cases, we have $w(\varphi, k) \leq t(k-1)^2 2^{n-t-1} + (k-1)2^{n-t} \leq tk^2 2^{n-t}$. $\square$

Now we are ready to prove Theorem 1. The idea of the proof is that any Gallai coloring of K_n can be treated as an extension of a coloring in $F(a,k)$ for some maximum a. Furthermore, we need to consider the largest 2-colored clique of the coloring on such a vertices.

2.3. **Proof of Theorem 1.** For each coloring $\varphi \in \Phi_{n\to k}$, let $A(\varphi)$ be a set of vertices of *maximum* size such that the restriction of φ to $K_n[A(\varphi)]$ forms a coloring in $F(|A(\varphi)|,k)$ (in case there is more than one choice for $A(\varphi)$, select one arbitrarily). Furthermore, let $M(\varphi)$ be a set of vertices of *maximum* size such that $K_n[M(\varphi)]$ is colored with at most 2 colors. Let $a := a(\varphi) = |A(\varphi)|$ and $m := m(\varphi) = |M(\varphi)|$. (In particular, a maximum 2-colored set in $F(|A(\varphi)|,k)$ has at most m vertices).

We say that a 2-coloring of $E(K_m)$ is *nearly monochromatic* if one of the colors is used at most $m^2/20$ times. Consider the following sets of colorings.

$$\begin{aligned}
\mathcal{C}_1 &= \left\{\varphi \in \Phi_{n\to k} \colon a(\varphi) < \frac{n}{6}\right\},\\
\mathcal{C}_2 &= \left\{\varphi \in \Phi_{n\to k} \colon n > m(\varphi) \geq \frac{n}{7} \text{ and } M(\varphi) \text{ is not nearly monochromatic}\right\},\\
\mathcal{C}_3 &= \left\{\varphi \in \Phi_{n\to k} \colon n > m(\varphi) \geq \frac{n}{7} \text{ and } M(\varphi) \text{ is nearly monochromatic}\right\} \text{ and}\\
\mathcal{C}_4 &= \left\{\varphi \in \Phi_{n\to k} \colon a(\varphi) \geq \frac{n}{6} \text{ and } m \leq \frac{n}{7}\right\}.
\end{aligned}$$

Note that $\Phi_{n\to k} \smallsetminus (\mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{C}_4)$ is the set of the k-colorings of $E(K_n)$ that use at most two of the colors (that is, $m = n$). Thus, we have $|\Phi_{n\to k} \smallsetminus (\mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{C}_4)| < \binom{k}{2}2^{\binom{n}{2}}$. To prove Theorem 1, it remains to show that $|\mathcal{C}_i| \leq 2^{\binom{n}{2}}o_n(1)$ for each $i \in [4]$. The main idea to bound $|\mathcal{C}_1|$ (resp. $|\mathcal{C}_2|$) is that although there are many options for how we choose and color $A(\varphi)$ (resp. $M(\varphi)$), those colorings can be extended to a coloring of K_n in few ways. The main gain while counting $|\mathcal{C}_1|$ comes from applying Lemma 11 many times, and for $|\mathcal{C}_2|$ it comes from applying Lemma 8 many times. On the other hand, to bound $|\mathcal{C}_3|$ (resp. $|\mathcal{C}_4|$) there are so few ways to color the set $M(\varphi)$ (resp. $A(\varphi)$) that even if we use the general bound on Lemma 6 to count the extensions of these colorings, we get few colorings in $\mathcal{C}_3$ (resp. $\mathcal{C}_4$). We use an ad-hoc argument in the case of $\mathcal{C}_3$ and use Lemma 10 in the case of $\mathcal{C}_4$.

Upper bound for $|\mathcal{C}_1|$. Fix an arbitrary ordering, say $(v_1,\ldots,v_n)$, of the vertices of K_n and, for each $j \in [n]$, let K_j be the graph induced by $\{v_1,\ldots,v_j\}$.

Let $s = \lceil n/2 \rceil$. We start with a Gallai k-coloring of K_s and count in how many ways it can be extended to a coloring of $E(K_n)$ that belongs to $\mathcal{C}_1$. By Corollary 7, there are less than $k^s 2^{\binom{s}{2}}$ such colorings of $E(K_s)$. Now, for each j from s to $n-1$, since we want to count the colorings in $\mathcal{C}_1$, keep only those colorings of $E(K_j)$ that do not have a set of $n/6$ vertices which induces a coloring in $F(n/6,k)$. As $n/6 \leq j/3$, they also do not have $j/3$ vertices that induce a coloring in $F(j/3,k)$. By Lemma 11 with j (in place of n in Lemma 11) and $t = n/150 \leq j/75$, we conclude that there are at most $tk^2 2^{j-t}$ extensions from K_j to K_{j+1}. Using that $2^{\binom{s}{2}} \prod_{j=s}^{n-1} 2^j = 2^{\binom{n}{2}}$,

$k \leq 2^{n/500}$ and n is large enough, we have

$$|\mathcal{C}_1| \leq k^s 2^{\binom{s}{2}} \left(\prod_{j=s}^{n-1} tk^2 2^{j-t} \right) \leq 2^{\binom{n}{2}} k \left(\frac{tk^3}{2^t} \right)^{\lfloor n/2 \rfloor} = 2^{\binom{n}{2}} k \left(\frac{nk^3}{150 \cdot 2^{n/150}} \right)^{\lfloor n/2 \rfloor} = 2^{\binom{n}{2}} o_n(1).$$

Upper bound for $|\mathcal{C}_2|$. For each m with $n/7 \leq m < n$, there are $\binom{n}{m} \leq n^{n-m}$ ways to choose a set S of m vertices as a candidate for $M(\varphi)$ and less than $\binom{k}{2} 2^{\binom{m}{2}}$ ways to color the edges induced by it. Consider only those 2-colorings of the edges in S that are not nearly monochromatic, that is, both colors are used more than $m^2/20$ times.

This time the counting of extensions of such colorings to K_n is done in a different way. Let $V(K_n) \setminus S = \{v_{m+1}, \dots, v_n\}$. We have to color all edges incident to those vertices.

For i varying from $m+1$ to n, first we color the edges from v_i to S and then we color the edges from v_i to $\{v_{m+1}, \dots, v_{i-1}\}$. Notice that, by the maximality of m, the edges from v_i to S must use one color different from the colors on S. Therefore, by Lemma 8 (with $s = m/60$), there are at most $km2^{m-m/150}$ ways to color those edges. Moreover, by Lemma 6 there are at most $k2^{i-m-1}$ ways to color the edges from v_i to $\{v_{m+1}, \dots, v_{i-1}\}$. Thus, as $k \leq 2^{n/4300}$ and n is large enough, we have

$$\begin{aligned}
|\mathcal{C}_2| &\leq \sum_{m=n/7}^{n-1} n^{n-m} \binom{k}{2} 2^{\binom{m}{2}} \left(\prod_{i=m}^{n-1} km2^{m-m/150} k2^{i-m} \right) \\
&\leq k^2 2^{\binom{n}{2}} \sum_{m=n/7}^{n-1} n^{n-m} \left(\frac{k^2 m}{2^{m/150}} \right)^{n-m} \\
&\leq k^2 2^{\binom{n}{2}} \sum_{m=n/7}^{n-1} \left(\frac{n^2 k^2}{2^{n/1050}} \right)^{n-m} \leq 2^{\binom{n}{2}} \frac{n^3 k^4}{2^{n/1050}} = 2^{\binom{n}{2}} o_n(1).
\end{aligned}$$

Upper bound for $|\mathcal{C}_3|$. For m with $n/7 \leq m < n$, we have $\binom{n}{m} < 2^n$ ways to choose a set S of m vertices. We give an upper bound for how many colorings $\varphi \in \mathcal{C}_3$ are such that $M(\varphi) = S$. First, there are $k(k-1) \leq k^2$ ways to choose the 2 colors for the edges in $M(\varphi)$ and to select one color to be the less frequent one of them, say blue. Let c be the number of blue edges in $M(\varphi)$. Since $M(\varphi)$ is nearly monochromatic, we have $c \leq m^2/20$. For $0 \leq c \leq m^2/20$, we have

$$\binom{\binom{m}{2}}{c} \leq \binom{m^2/2}{c} \leq \binom{m^2/2}{m^2/20} \leq (10e)^{m^2/20}.$$

Thus, as $m \geq n/7$ is large enough, the number of ways to color the edges induced by S is at most

$$k^2 \sum_{c=0}^{m^2/20} \binom{\binom{m}{2}}{c} \leq k^2 \left(\frac{m^2}{20} + 1 \right) (10e)^{m^2/20} \leq k^2 2^{m^2/4},$$

because $\log(10e) \leq 4.8$. This is already small enough that, to bound $|\mathcal{C}_3|$, we simply use Lemma 6 for each j from m to $n-1$ (to count in how many ways we can extend each coloring of S to K_n).

Thus, as $k \le 2^{n/200}$ and n is large enough, we have

$$\begin{aligned}|\mathcal{C}_3| &\le \sum_{m=n/7}^{n-1} 2^n k^2 2^{m^2/4} \prod_{j=m}^{n-1} k2^j \\ &\le \sum_{m=n/7}^{n-1} 2^{n+m^2/4} k^n \, 2^{\binom{n}{2}-\binom{m}{2}} \\ &\le 2^{\binom{n}{2}} \sum_{m=n/7}^{n-1} 2^{n+n\log k+m^2/4-\binom{m}{2}} \\ &\le 2^{\binom{n}{2}} n 2^{n+n^2/200+(n/7)^2/4-\binom{n/7}{2}} = 2^{\binom{n}{2}} o_n(1).\end{aligned}$$

Upper bound for $|\mathcal{C}_4|$. Our counting this time is similar to that for $|\mathcal{C}_3|$. For a such that $n/6 \le a \le n$, we have $\binom{n}{a} < 2^n$ ways to choose a set S of a vertices. Now, we bound the number of colorings $\varphi \in \mathcal{C}_4$ such that $A(\varphi) = S$. By the definition of $\mathcal{C}_4$ (as $6/7 < 0.9$) and Lemma 10, we know that for every $\varphi \in \mathcal{C}_4$, there are $f'(a,k) \le 2^{\binom{a}{2}-0.05a^2+(a+1)\log k}$ possibilities for the coloring of $A(\varphi)$. We then use Lemma 6 for each j from a to $n-1$ to count in how many ways we can extend each coloring of S to K_n. Thus, we obtain

$$\begin{aligned}|\mathcal{C}_4| &\le \sum_{a=n/6}^{n-1} 2^n 2^{\binom{a}{2}-0.05a^2+(a+1)\log k} \prod_{j=a}^{n-1} k2^j \\ &\le 2^{\binom{n}{2}} \sum_{a=n/6}^{n-1} 2^{n-0.05a^2+(a+1)\log k} k^n \\ &= 2^{\binom{n}{2}} \sum_{a=n/6}^{n-1} 2^{n-0.05a^2+(a+n+1)\log k}.\end{aligned}$$

As $k \le 2^{n/60}$, the function $h(a) = -0.05a^2 + a\log k + (n+1)\log k$ is decreasing on $n/6 \le a$. Furthermore, as $k \le 2^{n/900}$ and n is large enough, we have

$$n - 0.05a^2 + (a+n+1)\log k \le n - \frac{0.05}{36}n^2 + \left(\frac{7}{6}n + 1\right)\frac{n}{900} \le -n.$$

Therefore, we have $|\mathcal{C}_4| \le 2^{\binom{n}{2}} n 2^{-n} = 2^{\binom{n}{2}} o_n(1)$.

The proof of Theorem 1 is complete. □

§3. Concluding remarks

We considered the problem of finding the number of Gallai k-colorings of K_n and gave an asymptotically sharp bound for that number, even when k is substantially larger than n. Note that, since no Gallai coloring of K_n uses more than $n-1$ colors, we were originally considering only the case $3 \le k \le n-1$ and that case is completely solved by Theorem 1 with room to spare.

In Lemma 6, we gave an upper bound for $w(\varphi, k)$. Now, given any Gallai coloring of K_n, adding an extra vertex u and coloring all edges from u to K_n with the same color, we obtain a Gallai coloring of K_{n+1}. Therefore, $w(\varphi, k) \ge k$. This gives another proof that $c(n,k) \ge k^{n-1}$.

The discussion about the star colorings gives rise to another question: consider the smallest function $k_0(n)$ such that for $k > k_0(n)$ the number of star colorings is larger than the number of 2-colorings. Are there constants c and C such that for $k \leq ck_0(n)$ most Gallai colorings are 2-colorings while for $k \geq Ck_0(n)$ most are star colorings? Could we have a sharper threshold? If not, is there some other function $g(n)$ such that for $k > g(n)$ most Gallai colorings are star?

One may also ask if there are threshold functions $k_\ell(n)$ such that for $k > k_\ell(n)$ the number of Gallai colorings that use exactly ℓ colors is larger than the number of those that use exactly $\ell - 1$ colors.

Finally, one may also ask for an improvement to the (weak) general upper bound given by Corollary 7, so that it still works for every pair (n, k), regardless of their relative magnitudes. We fell that Corollary 7 can be significantly improved, because we conjecture that the number of colorings obtained by combining 2-colorings to star colorings (for example, start with a 2-coloring of some K_t and extend via monochromatic stars until obtaining n vertices) is significantly less than the sum of number of 2-coloring with the number of star colorings. But there maybe be colorings with a completely different structure that are also frequent when k is very large.

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