Additive Properties of a Pair of Sequences

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Abstract

Motivated by a question of Sárközy, we investigate sufficient conditions for existence of sets of natural numbers $A$ and $B$ such that the number of solutions of the equation $a + b = n$ where $a \in A$ and $b \in B$ is monotone increasing for $n > n_0$. We also examine sets $A$, $B$ with the property that, for every $n \geq 0$, the equation above has at most one solution, i.e., all pairwise sums are distinct.

1 Introduction

For a given set $A \subset \mathbb{N}_0$ of non-negative integers, here and throughout the paper, the counting function $A(n)$ is defined as the number of elements of $A$ not exceeding $n$, i.e., $A(n) = |A \cap \{0, 1, 2, \ldots, n\}|$. Consider the following functions

\[
\begin{align*}
    r(A, n) &= |\{(a_1, a_2) \in A \times A : a_1 + a_2 = n\}|, \\
    r_1(A, n) &= |\{(a_1, a_2) \in A \times A : a_1 + a_2 = n \text{ and } a_1 \leq a_2\}|, \\
    r_2(A, n) &= |\{(a_1, a_2) \in A \times A : a_1 + a_2 = n \text{ and } a_1 < a_2\}|.
\end{align*}
\]

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A well-studied problem concerning these functions is to determine necessary and sufficient conditions on \( A \) for their (eventual) monotonicity. Here and throughout the paper, monotonicity refers to monotonicity in \( n \). In other words, for what sets \( A \) we can find an \( n_0 \) such that \( r(A, n+1) \geq r(A, n) \) for all \( n > n_0 \)? Although the three functions look similar, and in fact \( |r(A, n) - 2r_2(A, n)| \leq 1 \) and \( |r_1(A, n) - r_2(A, n)| \leq 1 \), the (partial) answers to these questions may be quite different.

Erdős, Sárközy and Sós [3] proved that \( r(A, n) \) is eventually monotone increasing if and only if \( A \) contains all the positive integers from a certain point on. On the other hand, they obtained only a partial answer for \( r_1 \) and \( r_2 \). In particular, they proved that if

\[
\lim_{n \to +\infty} \frac{n - A(n)}{\log n} = +\infty
\]

then \( r_1(A, n) \) is not eventually monotone increasing. (This result was also obtained independently by Balasubramanian [1].)

Also, for \( r_2(A, n) \) they proved that if

\[
A(n) = o \left( \frac{n}{\log n} \right)
\]

then \( r_2(A, n) \) cannot be monotone increasing from a certain point on.

Motivated by these results, Sárközy asked the following question in his valuable survey of unsolved problems in number theory [8] (see Problem 4 in [8]).

**Problem 1.** If \( A, B \) are given infinite sets of non-negative integers, what can one say about the monotonicity of the number of solutions of the equation

\[
a + b = n, \ a \in A, \ b \in B
\]

We can naturally rephrase this question by defining the following function.

**Definition 2.** The *representation function* for two sets \( A, B \subset \mathbb{N}_0 \) is

\[
r(A, B, n) = |\{(a, b) \in A \times B : a + b = n\}|.
\]
The main goal of the present paper is to give some sufficient conditions on $A, B$ for the monotonicity of this function. This new representation function acts surprisingly different from the prequel. Our main result is as follows.

**Theorem 3.** For all $0 \leq \alpha, \beta < 1$, $1/2 < c_1, c_2 \leq 1$, there exist sets $A, B \subset \mathbb{N}_0$ such that $r(A, B, n)$ is monotone increasing in $n$:

$$\limsup_{n \to \infty} \frac{A(n)}{n^{c_1}} = \alpha; \quad \limsup_{n \to \infty} \frac{B(n)}{n^{c_2}} = \beta.$$ 

In the next sections we develop tools to approach Theorem 3 and prove some related results. Then we will return to the proof of Theorem 3.

## 2 co-Sidon Sets

Before proving Theorem 3, we introduce a generalized notion of Sidon sets and study some of its properties. Recall that a set $A \subset \mathbb{N}_0$ is called *Sidon* if $r_1(A, n) \leq 1$ for all $n \in \mathbb{N}$, i.e., the sums of unordered pairs of elements of $A$ are all distinct. We remark that it is possible to extend the notion of a Sidon set to a pair of sets in different ways. In this paper, we consider the following generalization.

**Definition 4.** Two sets $A, B \subset \mathbb{N}_0$ are called *co-Sidon* if $r(A, B, n) \leq 1$ for all $n \in \mathbb{N}_0$, i.e., the sums $a + b$ are distinct for all $(a, b) \in A \times B$.

Note that if $A, B$ are co-Sidon then $|A \cap B| \leq 1$.

For sets $A$ and $B$ of integers we denote their *sum set* by $A + B = \{a + b : a \in A, b \in B\}$. For simplicity if the set $B$ is a single element $b$ we denote their sum set by $A + b = A + B$.

When $A, B$ are finite sets, we prove a simple but sharp result about $|A|, |B|$.

**Proposition 5.** If $A, B \subset \{0, 1, 2, \ldots, n\}$ are co-Sidon, then

$$\min \{|A|, |B|\} \leq \lfloor \sqrt{2n} \rfloor.$$ 

Furthermore, equality can be obtained for infinitely many values of $n$. 

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Proof. Since $A$ and $B$ are finite (and co-Sidon) we have $|A + B| = |A||B|$. Without loss of generality assume $|A| \leq |B|$. Then, $|A|^2 \leq |A + B|$.

Clearly for an element $c \in A + B$ we have $0 \leq c \leq 2n$. However, either $0$ or $2n$ is not an element of $A + B$, otherwise we would have $0, n \in A \cap B$ and there would be two distinct solutions to $a + b = n$ with $a \in A$ and $b \in B$. Thus, $|A + B| \leq 2n$ which yields $|A| \leq \lfloor \sqrt{2n} \rfloor$ and the upper-bound is established.

To see that the upper bound is best possible for infinitely many $n$, consider the following construction for $A$ and $B$. Let $m \in \mathbb{N}$ be fixed and define

$$A := \{0, m, 2m, \ldots, (2m - 1)m\}$$

and

$$B := \{0, 1, 2, \ldots, m - 1, 2m^2, 2m^2 + 1, 2m^2 + 2, \ldots, 2m^2 + m - 1\}.$$

Note that $|A| = |B| = 2m$ and $A + B = \{0, 1, \ldots, 4m^2 - 1\}$. Therefore $A$ and $B$ are co-Sidon. As $A, B \subseteq \{0, 1, 2, \ldots, 2m^2 + m - 1\}$, we can take $n = 2m^2 + m - 1$. This gives

$$2m = \sqrt{4m^2} \leq \sqrt{4m^2 + 2m - 2} = \sqrt{2n} < \sqrt{4m^2 + 4m + 1} = 2m + 1.$$  

Hence $\min \{|A|, |B|\} = 2m = \lfloor \sqrt{2n} \rfloor$. As the choice of $m$ was arbitrary, there are infinitely many $n$ for which we can reach the upper bound in the statement of the theorem.

It is worth to compare the above result to the following theorem of Erdős and Turán [4] on finite Sidon sets.

**Theorem 6.** There is an absolute positive constant $c$ such that if $n \in \mathbb{N}$ and $A \subset \{1, 2, \ldots, n\}$ is a Sidon set, then $|A| < n^{1/2} + cn^{1/4}$.

On the other hand, the best known constructions give Sidon sets of size $n^{1/2}$ for infinitely many $n$ (see e.g. [5, 7] for details). The reduction of this gap is a well-known hard problem.

We consider now the case where $A, B$ are infinite co-Sidon. Defining $A_n = A \cap \{0, 1, \ldots, n\}$ and $B_n = B \cap \{0, 1, \ldots, n\}$, we have that $A_n, B_n$ are co-Sidon. So, by Theorem 5, for any $n$ we have

$$\min \{A(n), B(n)\}/\sqrt{n} = \min \{|A_n|, |B_n|\}/\sqrt{n} \leq \lfloor \sqrt{2n} \rfloor /\sqrt{n} \leq \sqrt{2}.$$
A simple example shows that we can come close to achieving this bound.

**Construction 7.** Let $A$ be the set of integers which can be written in the form $\sum_{i=0}^{k} \alpha_i 2^{2i}$ where $\alpha_i \in \{0, 1\}$ and $k \in \mathbb{N}$. Let $B$ be the set of integers which can be written in the form $\sum_{i=0}^{k} \alpha_i 2^{2i+1}$ where $\alpha_i \in \{0, 1\}$ and $k \in \mathbb{N}$. It is clear that $A$ and $B$ are co-Sidon and $A + B = \mathbb{N}_0$. It can easily be verified that

\[
\lim_{n \to \infty} \frac{\min\{A(n), B(n)\}}{\sqrt{n}} = \sqrt{2}/2.
\]

Comparing this with the following result of Erdős (see [9, 5]), we conclude that infinite Sidon sets and infinite co-Sidon sets also behave differently. In general, we have more freedom when working with co-Sidon sets.

**Theorem 8.** There is an absolute, positive constant $c$ such that for any infinite Sidon set $A \subset \mathbb{N}$ we have

\[
\lim_{n \to \infty} \frac{A(n)}{\sqrt{n/\log n}} < c.
\]

It is also worth mentioning the following theorem of Krückeberg [6] for infinite Sidon sets.

**Theorem 9.** There is a Sidon set $A \subset \mathbb{N}$ such that

\[
\limsup_{n \to \infty} \frac{A(n)}{\sqrt{n}} \geq \sqrt{2}/2.
\]
The following definition will be useful for us.

**Definition 10.** We call sets \( A, B \subset \mathbb{N}_0 \) perfect if the sum set \( A + B \) is an interval (possibly unbounded) of consecutive integers.

The next proposition will be helpful in building new perfect co-Sidon sets from other co-Sidon sets.

**Proposition 11.** Let \( A, B \subset \mathbb{N}_0 \) be finite perfect co-Sidon sets. Let \( c = \max (A) + \max (B) - \min (A) - \min (B) + 1 \). Then for any \( k \in \mathbb{N}_0 \), the sets \( A \) and \( C = \bigcup_{i=0}^{k} (B + ic) \) are perfect co-Sidon.

**Proof.** Let \( r = \min (A) + \min (B) \). By assumption, \( A + B = \{r, r + 1, \ldots, c + r - 1\} \). For each \( i \), the sets \( A \) and \( B + ic \) are co-Sidon. Furthermore, the sets

\[
A + (B + c) = \{c + r, c + r + 1, \ldots, 2c + r - 1\}
A + (B + 2c) = \{2c + r, 2c + r + 2, \ldots, 3c + r - 1\}
\vdots
A + (B + kc) = \{kc + r, kc + r + 1, \ldots, (k+1)c + r - 1\}
\]

are all pairwise disjoint consecutive intervals. Therefore \( A \) and \( \bigcup_{i=0}^{k} (B + ic) \) are perfect co-Sidon with sum set \( \{r, r + 1, \ldots, (k+1)c + r - 1\} \). \( \square \)

Clearly the proposition also holds for \( C = \bigcup_{i=0}^{\infty} (B + ic) \).

Next we characterize all infinite perfect co-Sidon sets \( A, B \subset \mathbb{N}_0 \) using the mixed radix representation. Note that both the co-Sidon and perfect properties are invariant under translation of each of the sets (i.e. addition or subtraction by a constant), so without loss of generality we may assume \( 0 \in A \cap B \).

**Theorem 12.** Let \( A, B \subset \mathbb{N}_0 \) be infinite, such that \( 0 \in A \cap B \). Then \( A, B \) are perfect co-Sidon if and only if there exists an infinite sequence of integers \( (k_i)_{i=1}^{\infty} \) such that \( \forall i, k_i \geq 2 \) and (up to an exchange of \( A \) and \( B \)),

\[
A = \left\{ \sum_{i=1}^{\infty} k_1 k_2 \ldots k_{2i-2} a_{2i-1} : \forall j, 0 \leq a_{2j-1} < k_{2j-1}, \text{ finitely many } a_{2i-1} \text{ non-zero} \right\}
\]
and
\[ B = \left\{ \sum_{i=1}^{\infty} k_1 k_2 \ldots k_{i-1} a_{2i} : \forall j, 0 \leq a_{2j} < k_{2j}, \text{ finitely many } a_{2i} \text{ non-zero} \right\}. \]

**Proof.** A sum of the form \( \sum_{i=1}^{\infty} k_1 k_2 \ldots k_{i-1} a_i \) where \( 0 \leq a_j < k_j \), and only finitely many \( a_i \) are non-zero, is precisely the so-called mixed-radix representation with bases \((k_1, k_2, \ldots, k_i, \ldots)\). Thus the base \( r \) representation is the special case where \( k_i = r \) for all \( i \). For any sequence \((k_i)_{i=1}^{\infty}\) of integers with \( k_i \geq 2 \), every non-negative integer is uniquely representable with bases \((k_i)\).

Let \((k_i)_{i=1}^{\infty}\) be a sequence of integers such that \( \forall i, k_i \geq 2 \). Suppose \( A \) and \( B \) are of the form determined by the bases \( k_i \) as above. As every non-negative integer is uniquely representable with bases \((k_i)\), \( A \) and \( B \) are co-Sidon. Also observe that
\[ A + B = \left\{ \sum_{i=1}^{\infty} k_1 k_2 \ldots k_{i-1} a_i : \forall j, 0 \leq a_j < k_j, \text{ finitely many } a_i \text{ non-zero} \right\}. \]

Thus \( A + B = \mathbb{N}_0 \) and therefore \( A \) and \( B \) are perfect.

Now assume that \( A, B \) are perfect co-Sidon. Unless \( A = B = \{0\} \), we can assume without loss of generality that \( 1 \in A \). To show that \( A, B \) are of the required form, we need to construct a sequence of base elements \((k_i)_{i \in \mathbb{N}}\) that represents \( A \) and \( B \) as in the statement of the theorem.

Our construction of the integers \( k_i \) is recursive. Let \( k_0 = 1 \). For \( t \geq 1 \) define \( c_t = k_t - 1 \) and let
\[
k_t = \begin{cases} 
\max \{a : \{c_t, 2c_t, \ldots, (a-1)c_t\} \subset A\}, & \text{if } t \text{ is odd} \\
\max \{b : \{c_t, 2c_t, \ldots, (b-1)c_t\} \subset B\}, & \text{if } t \text{ is even}
\end{cases}
\]

Note that \( \forall t > 0, k_t < \infty \). Otherwise, one of \( A \) or \( B \) contains an infinite arithmetic progression, whose consecutive terms differ by \( c_t \). But as they are co-Sidon, this implies that the other set is finite (in fact of order at most \( c_t \)), a contradiction.

Now define two families of sets. Let \( A_0 = B_0 = \{0\} \) and for each \( t \geq 1 \),
\[
A_t = \left\{ \sum_{i=1}^{t} k_1 k_2 \ldots k_{i-1} a_i : \forall j, 0 \leq a_j < k_j \text{ and } a_{2j} = 0 \right\}
\]
and

\[ B_t = \left\{ \sum_{i=1}^{t} k_1 k_2 \ldots k_{i-1} b_i : \forall j, 0 \leq b_j < k_j \text{ and } b_{2j-1} = 0 \right\}. \]

Note that for all \( j \), \( A_{2j} = A_{2j-1} \) and \( B_{2j-1} = B_{2j-2} \). Let \( A^* = \bigcup_{t=0}^{\infty} A_t \) and \( B^* = \bigcup_{t=0}^{\infty} B_t \). It only remains to prove that \( A = A^* \) and \( B = B^* \). We will use the following claim.

**Claim 13.** For all \( t \geq 0 \)

\[
\begin{align*}
A \cap \{0, 1, \ldots, k_1 \ldots k_t - 1\} &= A_t \\
B \cap \{0, 1, \ldots, k_1 \ldots k_t - 1\} &= B_t.
\end{align*}
\]

**Proof.** Suppose not and let \( t \) be minimal such that the claim does not hold. Thus there must exist an \( x \in \mathbb{N} \) such that either \( x \in (A \cap \{0, 1, \ldots, k_1 k_2 \ldots k_t - 1\}) \Delta A_t \) or \( x \in (B \cap \{0, 1, \ldots, k_1 k_2 \ldots k_t - 1\}) \Delta B_t \), where \( \Delta \) denotes the symmetric difference of sets. Pick a minimal such \( x \). Let us assume that \( t \) is odd and \( t \geq 3 \); the proof is trivial for \( t = 0 \) or \( t = 1 \) and similar when \( t \geq 2 \) is even. As \( t \) is odd (and minimal) \( B_t = B_{t-1} = B \cap \{0, 1, \ldots, k_1 \ldots k_{t-1} - 1\} \subset B \cap \{0, 1, \ldots, k_1 \ldots k_{t-1} - 1\} \), thus \( B_t \setminus (B \cap \{0, 1, \ldots, k_1 \ldots k_{t-1} - 1\}) \) is empty.

Now write

\[ x = \sum_{i=1}^{t} k_1 k_2 \ldots k_{i-1} a_i \]

in the mixed-radix representation with bases \((k_i)_{i=1}^{\infty}\). Set

\[ z = \sum_{i=0}^{\left\lfloor \frac{t}{2} \right\rfloor} k_1 \ldots k_{2i} a_{2i+1} \]

and

\[ w = \sum_{i=1}^{\left\lfloor \frac{t}{2} \right\rfloor} k_1 \ldots k_{2i-1} a_{2i}. \]

By definition, \( z \in A_t \), \( w \in B_t = B_{t-1} \) and \( x = z + w \). By the minimality of \( t \), \( B_{t-1} \subset B \), thus \( w \in B \). We now distinguish the remaining three cases.
(i) Suppose \( x \in (A \cap \{0, 1, \ldots, k_1 \cdots k_t - 1\}) \setminus A_t \). Since \( x \notin A_t \), we have \( x \neq z \), thus \( z \in A \) by minimality of \( x \). Now we have that \( x, z \in A \) and \( 0, w \in B \). But \( x + 0 = z + w \), contradicting the fact that \( A \) and \( B \) are co-Sidon.

(ii) Suppose \( x \in A_t \setminus (A \cap \{0, 1, \ldots, k_1 \cdots k_t - 1\}) \). As \( A + B = \mathbb{N}_0 \), we can write \( x = a + b \) with \( a \in A, b \in B \). Note that \( x \leq k_1k_2 \cdots k_t - 1 \) and this implies \( x \notin A \). In particular, \( x \neq a \). We claim that \( x = b \). If not, then \( 0 < a, b < x \) and the minimality of \( x \) implies that \( a \in A_t \) and \( b \in B_t \). But \( a + b = x \in A_t \), which contradicts the definition of \( A_t \) and \( B_t \). Thus we may suppose \( x = b \), i.e., \( x \in A_t \cap B_t \).

For \( 0 \leq i \leq \left\lfloor \frac{t}{2} \right\rfloor - 1 \), define
\[
\alpha_{2i+1} = \begin{cases} 
    k_{2i+1} - a_{2i+1} & \text{if } a_{2i+1} > 0 \\
    0 & \text{if } a_{2i+1} = 0
\end{cases}
\]
and
\[
\beta_{2i+2} = \begin{cases} 
    0 & \text{if } \alpha_{2i+1} = 0 \\
    1 & \text{if } \alpha_{2i+1} > 0.
\end{cases}
\]
Let
\[
u = (\alpha_{t-1}0\alpha_{t-4} \cdots \alpha_3 - \alpha_1)(k_i) = \sum_{i=0}^{\left\lfloor \frac{t}{2} \right\rfloor - 1} k_1 \cdots k_{2i+1} \alpha_{2i+1} \in A_t - 2,
\]
\[
v = (\beta_{t-1}0\beta_{t-3}0 \cdots \beta_20)(k_i) = \sum_{i=1}^{\left\lfloor \frac{t}{2} \right\rfloor} k_1 \cdots k_{2i-1} \beta_{2i}.
\]
By definition of \( k_t \), \( a_t \prod_{i=0}^{t-1} k_i \in A \) and by minimality of \( t \), we have \( u \in A \) and \( v \in B \). Clearly, \( u \neq a_t \prod_{i=0}^{t-1} k_i \). But \( u + x = a_t \prod_{i=0}^{t-1} k_i + v \), contradicting the fact that \( A \) and \( B \) are co-Sidon.

(iii) Suppose \( x \in (B \cap \{0, 1, \ldots, k_1 \cdots k_t - 1\}) \setminus B_t \). Clearly \( x \notin A \), otherwise \( 0, x \in A \cap B \) which contradicts \( A, B \) being co-Sidon. Also \( x \notin A_t \), otherwise \( x \in A_t \cap B \) and we can continue as at the end of case (ii). Thus \( x \neq z \), this implies \( z \in A \) by the minimality of \( x \). Also \( w \in B_t \) implies \( x \neq w \). Now \( 0 + x = z + w \), with \( 0, z \in A \) and \( x, w \in B \) contradicting the fact that \( A \) and \( B \) are co-Sidon. \( \square \)
To complete the proof of the theorem, we must show $\forall t > 0, k_t \geq 2$. Suppose that $k_{t_0} = 1$. That is, $c_{t_0} = k_1 k_2 \cdots k_{t_0 - 1}$ is in neither $A$ nor $B$. But then as $A$ and $B$ are perfect co-Sidon, there exist $a \in A$ and $b \in B$ such that $a + b = c_{t_0}$. By assumption, $a, b < c_{t_0}$. But clearly $(a, b) \notin A_{t_0} \times B_{t_0}$ as $A_{t_0} + B_{t_0} \subset \{0, 1, \ldots, c_{t_0} - 1\}$ contradicting Claim 13.

Theorem 12 allows us to make a useful observation about the structure of perfect co-Sidon sets.

**Corollary 14.** If $A$ and $B$ are infinite perfect co-Sidon sets then for all $m \in \mathbb{N}$ there are infinitely many $n \in \mathbb{N}$ such that $\{n, n + 1, \ldots, 2n + m\} \cap A = \emptyset$.

**Proof.** As the statement remains true when we translate $A$ or $B$, it suffices to prove it for $A$ and $B$ with $0 \in A \cap B$. There exists an infinite sequence of integers $(k_i) \forall i, k_i \geq 2$ such that $A$ and $B$ are represented by the bases $k_i$ as in Theorem 12. Fix $m \in \mathbb{N}$ and let $t$ be such that $2 \prod_{i=0}^{t-1} k_i - 3 \geq m$ and $(k_t - 1) \prod_{i=0}^{t-1} k_i \in A$. Then by Theorem 12 the next element in $A$ is exactly $\prod_{i=0}^{t+1} k_i$. Let $n = (k_t - 1) \prod_{i=0}^{t-1} k_i + 1$. Now

$$\prod_{i=0}^{t+1} k_i = k_{t+1} \{(k_t - 1) + 1\} \prod_{i=0}^{t-1} k_i$$

$$\geq 2 \left\{ n - 1 + \prod_{i=0}^{t-1} k_i \right\}$$

$$\geq 2n - 2 + m + 3 = 2n + m + 1.$$

Thus $\{n, n + 1, \ldots, 2n + m\} \cap A = \emptyset$. Since $A$ is infinite, it follows that for every $m$ there are infinitely many such $n$.

Of course, the claim also holds for $B$.

It is natural to ask whether all co-Sidon sets $A, B$ are subsets of perfect co-Sidon sets $A^*, B^*$. The answer turns out to be no as the following proposition shows.

**Proposition 15.** The sets $A = \{2^k : k \in \mathbb{N}, k \geq 9\}$ and $B = \{3^l : l \in \mathbb{N}, l \geq 9\}$ are co-Sidon and there are no perfect co-Sidon sets $A^*, B^*$ such that $A \subseteq A^*$ and $B \subseteq B^*$.
Proof. The Diophantine equation $2^k + 3^l = 2^m + 3^n$ with $k < m$ and $l > n$ has only five solutions (see [10]); all have exponents less than 9. This implies that $A$ and $B$ are co-Sidon.

Note that, for all $n \geq 2^9$, $A$ contains numbers between $n$ and $2n$. That is, for all $n$, \( A \cap \{n, n+1, \ldots, 2n\} \neq \emptyset \). However, if \( A^* \) and \( B^* \) are perfect co-Sidon sets such that $A \subset A^*$ and $B \subset B^*$, then according to Corollary 14 there is an $n$ with $A^* \cap \{n, n+1, \ldots, 2n+m\} = \emptyset$.

\[ \square \]

3 Representation Function

We seek to provide sufficient conditions on $A$ and $B$ so that the representation function $r(A, B, n) = |\{(a, b) \in A \times B : a + b = n\}|$ is (eventually) monotone increasing. For $C \subset \mathbb{N}_0$ let us denote its complement $\overline{C} = \mathbb{N}_0 \setminus C$.

It is easy to see that if either $A$ or $\overline{A}$ is finite and either $B$ or $\overline{B}$ is finite then $r(A, B, n)$ is eventually monotone. To see this, if $\overline{A}$ and $B$ are finite, then for all $n > \max(\overline{A}) + \max(B)$ we have that $b \in B$ implies $n - b \in A$ and thus $r(A, B, n) = |B|$. Also, if $\overline{A}$ and $\overline{B}$ are finite, then for all $n > \max(\overline{A}) + \max(\overline{B})$ we have $r(A, B, n) = n + 1 - |\overline{A}| - |\overline{B}|$. Finally, if $A$ and $B$ are both finite then it is obvious that $r(A, B, n)$ is eventually monotone. So the study is non-trivial only in the case when $A$ and $\overline{A}$ are both infinite.

**Proposition 16.** Let $A, B \subset \mathbb{N}_0$ be infinite perfect co-Sidon sets such that $A + B = \mathbb{N}_0$. Then, for any $A' \subset A$ and $B' \subset B$, the representation function $r(A + B', B + A', n)$ is monotone increasing.

Proof. Note that

$$r(A + B', B + A', n) = r\left(\bigcup_{b \in B'} A + b, \bigcup_{a \in A'} B + a, n\right)$$

$$= \sum_{a \in A', b \in B'} r(A + b, B + a, n)$$

The second equality holds because the unions are disjoint.
From $A + B = \mathbb{N}_0$ it follows that $(A + b) + (B + a) = \mathbb{N}_0 + a + b$ and thus each summand is

$$r(A+b,B+a,n) = \begin{cases} 0 & \text{if } n < a + b, \\ 1 & \text{if } n \geq a + b. \end{cases}$$

Therefore, the representation function $r(A + B', B + A', n)$ is monotone increasing.

It follows from Theorem 12 that sets $A$ and $B$ which are infinite perfect co-Sidon exist. Since the subsets in Proposition 16 are arbitrary, we can construct many sets $A$ and $B$ such that $r(A, B, n)$ is monotone increasing.

The next theorem allows us to choose sets $A$ and $B$ whose representation function is monotone and increasing and whose counting functions $A(n)$ and $B(n)$ grow at a controlled rate.

**Theorem 17.** Let $A, B \subset \mathbb{N}_0$ be infinite perfect co-Sidon such that $A + B = \mathbb{N}_0$. Let $f : \mathbb{N}_0 \to \mathbb{R}$ be such that $A(n) \leq f(n)$ and for every $M > 0$ there exists $n_0$ such that for $n > n_0$ we have $f(n) < n + 1 - MA(n)$. Then there exists a $B' \subseteq B$ such that

$$(A + B')(n) \leq f(n) \text{ for all } n \in \mathbb{N}_0$$

and

$$(A + B')(n) \geq f(n) - A(n) \text{ for infinitely many } n \in \mathbb{N}_0.$$ 

**Proof.** Let $A$ and $B$ be as in the statement and write $B = \{b_0 < b_1 < \ldots\}$. By assumption, $b_0 = 0$. Let us construct $B' \subseteq B$ greedily as follows: set $B'_0 = \{0\}$ and for $i > 0$ let

$$B'_{i+1} = \begin{cases} B'_i \cup \{b_{i+1}\} & \text{if } (A + (B'_i \cup \{b_{i+1}\}))(n) \leq f_A(n) \text{ for all } n \in \mathbb{N}_0, \\ B'_i & \text{otherwise.} \end{cases}$$

Then let $B' = \bigcup_{i=0}^\infty B'_i$. We claim that this $B'$ satisfies the conditions of the theorem. By the construction,

$$(A + B')(n) \leq f(n) \text{ for all } n \in \mathbb{N}_0.$$ 

To prove that the other inequality holds for infinitely many values of $n$, we first need to show that $B \setminus B'$ is infinite. Suppose that $B \setminus B'$ is finite, and
let $M = |B \setminus B'|$. Since $A + B' \setminus B' = \cup_{b \in B \setminus B'} (A + b)$ we have $(A + B \setminus B')(n) \leq MA(n)$ for every $n$. Now, clearly

$$
\bigcup_{b \in B'} (A + b) = \mathbb{N}_0 \setminus \left( \bigcup_{b \in B \setminus B'} (A + b) \right).
$$

It follows that $(A + B')(n) = n + 1 - (A + (B \setminus B'))(n) \geq n + 1 - MA(n)$ for all $n$. But, for large enough $n$, we have $n + 1 - MA(n) > f(n)$. Then, for large enough $n$ we would have $(A + B')(n) > f(n)$, which contradicts the construction of $B'$. Hence $B \setminus B'$ is infinite.

Therefore, for infinitely many $i$, we have $b_{i+1} \notin B'$. For such an $i$ we have $B_{i+1}' = B_i'$. Therefore, by definition of $B_{i+1}'$, there exists $n_{i+1}$ such that $(A + B_i' \cup \{b_{i+1}\})(n_{i+1}) > f(n_{i+1})$. Note that $n_{i+1} \geq b_{i+1}$, because for all $n < b_{i+1}$,

$$(A + B_i' \cup \{b_{i+1}\})(n) = (A + B_i')(n) \leq f_A(n).$$

Therefore there are infinitely many $n$ such that,

$$(A + B')(n) \geq (A + B_i')(n) \geq f(n) - A(n).$$

\[\square\]

Our main theorem follows as a corollary of Theorem 17. We restate it here for easy reference:

**Theorem 3.** For all $0 \leq \alpha, \beta < 1$, $1/2 < c_1, c_2 \leq 1$, there exist sets $A, B \subset \mathbb{N}_0$ such that $r(A, B, n)$ is monotone increasing in $n$;

$$
\limsup_{n \to \infty} \frac{A(n)}{n^{c_1}} = \alpha; \quad \limsup_{n \to \infty} \frac{B(n)}{n^{c_2}} = \beta.
$$

**Proof.** Suppose we are given constants $0 \leq \alpha < 1$ and $1/2 < c_1 \leq 1$. Let $A_0$, $B_0$ be perfect co-Sidon sets such that $A_0(n) = \Theta(n^{1/2})$, $B_0(n) = \Theta(n^{1/2})$ (e.g. Construction 7.) Let $f(n) = \alpha n^{c_1} + d$ where $d$ is a constant large enough such that $f(n) \geq A_0(n)$ for all $n$. Clearly for all $m > 0$ there exists an $n_0$ such that for $n > n_0$, $f(n) < n + 1 - mA_0(n)$. By Theorem 17, there is a $B' \subset B_0$.
such that \((A_0 + B')(n) \leq f(n)\) for all \(n\) and \((A_0 + B')(n) \geq f(n) - A_0(n)\) for infinitely many \(n\). Set \(A = A_0 + B'\). Then

\[
\alpha = \lim_{n \to \infty} \frac{f(n)}{n^{c_1}} \geq \limsup_{n \to \infty} \frac{A(n)}{n^{c_1}} \geq \lim_{n \to \infty} \frac{f(n) - A_0(n)}{n^{c_1}} = \alpha.
\]

We can construct \(B\) in the same manner. By Proposition 16, the representation function \(r(A, B, n)\) is monotone increasing. \(\square\)

By modifying the previous two proofs, we can restate Theorem 3 with either (or both) limit superiors replaced with limit inferiors. The details are left to the interested reader. Theorem 3 gives a strong answer about the densities of sets \(A\) and \(B\) with monotone representation function \(r(A, B, n)\).

When \(c_1 = c_2 = 1\) and \(\alpha, \beta \in \mathbb{Q}\) we can restate Theorem 3 by replacing the limit superiors with standard limits.

**Theorem 18.** For all rational \(0 \leq \alpha, \beta \leq 1\), there exist sets \(A, B \subset \mathbb{N}_0\) such that \(A\) has density \(\alpha\), \(B\) has density \(\beta\) and \(r(A, B, n)\) is monotone increasing in \(n\).

**Proof.** We construct \(A\) and \(B\) using mixed radix representation to describe its elements. Write \(\alpha = p_1/q_1\) and \(\beta = p_2/q_2\) where \(p_i, q_i \in \mathbb{N}\). Set \(k_1 = q_1\), \(k_2 = q_2\) and \(k_i = 2\) for all \(i > 2\). Let \(A_0\) be the set of all integers that can be written in the form

\[
\sum_{i=0}^{k} k_1 k_2 \cdots k_{2i} a_{2i+1}
\]

where for each \(i\), \(0 \leq a_{2i+1} < k_{2i+1}\). Similarly let \(B_0\) be the set of all integers that can be written in the form

\[
\sum_{i=1}^{k} k_1 k_2 \cdots k_{2i-1} b_{2i}
\]

where for each \(i\), \(0 \leq b_{2i} < k_{2i}\). Note that \(A_0\) and \(B_0\) are perfect co-Sidon.

Let \(A'\) be the subset of \(A_0\) consisting of all those integers whose \(k_1\)-digit (in the mixed radix representation) lies in the set \(\{0, 1, \ldots, p_1 - 1\}\). As \(p_1 \leq q_1\) we must have \(p_1 - 1 \leq k_1 - 1\). Thus \(A'\) is well-defined. Then \(B = A' + B_0\) is
the set of all numbers whose $k_1$-digit lies in \( \{ 0, \ldots, p_1 - 1 \} \) that is, $B$ consists of the numbers congruent to $0, 1, \ldots, p_1 - 1 \pmod{q_1}$. The density of this set is clearly $p_1/q_1$.

Similarly, let $B'$ be the subset of $B_0$ consisting of all those integers whose $k_2$-digit (in the mixed radix representation) lies in the set $\{ 0, 1, \ldots, p_2 - 1 \}$. Again as $p_2 \leq q_2$ we have $p_2 - 1 \leq k_2 - 1$ so $B'$ is also well-defined. A similar argument holds when we are considering $A = A_0 + B'$. Here, $A$ is the set of numbers whose $k_2$-digit is in $\{ 0, 1, \ldots, p_2 - 1 \}$. Thus $A$ consists exactly of the numbers less than or equal to $(p_2 - 1)q_1 \pmod{q_1q_2}$. This follows as the base of the first digit is $q_1$. Again it is clear that $A$ has density $(p_2q_1)/(q_1q_2) = p_2/q_2$.

By Proposition 16, $r(A, B, n)$ is monotone increasing. \hfill \Box

Finally, we determine for which sets $A$, $B$ the representation function $r(A, B, n)$ is eventually strictly increasing. The corresponding question for a single set has been considered by Chen and Tang [2] who discuss when the functions $r$, $r_1$, $r_2$ are strictly increasing. When considering two sets and the function $r$, the problem turns out to be easy.

**Proposition 19.** Let $A, B \subset \mathbb{N}_0$, then the representation function $r(A, B, n)$ is eventually strictly monotone increasing if and only if $\overline{A}$ and $\overline{B}$ are finite.

**Proof.** First, let us assume that $r(A, B, n)$ is eventually strictly increasing. We will use the trivial identity that

$$n + 1 = r(\mathbb{N}_0, \mathbb{N}_0, n) = r(A, B, n) + r(\overline{A}, B, n) + r(A, \overline{B}, n) + r(\overline{A}, \overline{B}, n),$$

which is equivalent to

$$n + 1 - r(A, B, n) = r(\overline{A}, B, n) + r(A, \overline{B}, n) + r(\overline{A}, \overline{B}, n).$$

In the last identity the left hand side is bounded, since we assumed that $r(A, B, n)$ is eventually strictly increasing. Thus so is the right hand side. Hence $r(\overline{A}, B, n)$, $r(A, \overline{B}, n)$ and $r(\overline{A}, \overline{B}, n)$ are all bounded. From this it follows that $r(\overline{A}, \mathbb{N}_0, n) = r(\overline{A}, B, n) + r(\overline{A}, \overline{B}, n)$ and $r(\mathbb{N}_0, \overline{B}, n) = r(A, \overline{B}, n) + r(\overline{A}, \overline{B}, n)$ are bounded. Thus $\overline{A}$ and $\overline{B}$ must be finite.
Now we assume that $\overline{A}$ and $\overline{B}$ are finite. For any $n > \max(\overline{A}) + \max(\overline{B})$ we know that $a \in \overline{A}$ implies $n - a \notin \overline{B}$ and vice versa, so we can write

$$r(A, B, n) = n + 1 - |\overline{A}| - |\overline{B}| < n + 2 - |\overline{A}| - |\overline{B}| = r(A, B, n + 1)$$

Thus for $n > \max(\overline{A}) + \max(\overline{B})$ the representation function is strictly increasing.

\[\square\]

4 Open Problems

A far-reaching goal would be to completely characterize co-Sidon sets. Which co-Sidon sets are subsets of some perfect co-Sidon sets? Are two random sets likely to be co-Sidon?

Can we completely characterize sets $A, B$ whose representation function is monotone increasing? Are there constructions that do not come from perfect co-Sidon sets?

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References


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